# Decomposing life insurance liabilities into risk factors

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**Abstract**

Life insurance liabilities are influenced by various sources of risk such as equity, interest rate, and mortality risk. Although it is common to measure the total risk of life insurance liabilities via advanced stochastic models, it is not clear how to allocate the randomness to those different sources of risk – a question of great practical relevance in view of risk management and product design. In this paper, we first derive properties we posit a meaningful risk decomposition should satisfy and show that each conventional approach from literature contradicts at least one of these properties. Then we propose a novel decomposition method primarily motivated by the martingale representation theorem, and show that this method actually satisfies all the meaningful risk decomposition properties. With the help of the Clark-Ocone formula from Malliavin calculus and Itô's lemma for diffusion processes, we derive explicit formulas for calculating the decomposition of relatively general life insurance liabilities. The proposed decomposition method is applied to the discounted payoff of a Guaranteed Minimum Death Benefit in order to determine equity, interest, and mortality risks, demonstrating the method's applicability and usefulness.

*Keywords*: Life insurance liabilities, risk decomposition, risk factors, risk management, stochastic modeling of financial and mortality risk

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# 1 Introduction

Decomposing insurance liabilities into risk factors associated with different sources of risk is a problem of great practical significance, particularly in the life sector, in view of risk management, product design, and capital regulation.[1] The primary contributions of this paper are twofold: On the one hand, we introduce properties for a *meaningful risk decomposition* and show that all decomposition methods proposed in literature suffer shortcomings in view of these properties. On the other hand, we propose a novel decomposition approach based on martingale representation, labeled *MRT decomposition*, and show that it satisfies all the meaningful risk decomposition properties. We discuss the calculation of the MRT decomposition in a relatively general life insurance setting with an arbitrary (finite) insurance portfolio, where the homogeneous group of insured is modeled by a counting process, and the (systematic) sources of risk are driven by a finite-dimensional Brownian motion. We derive explicit formulas in terms of Malliavin derivatives in the general case and in terms of derivatives of conditional expectations in the Markov case. Moreover, we provide detailed example calculations in the context of a Variable Annuity contract with a Guaranteed Minimum Death Benefit (GMDB). In particular, we illustrate that for moderate or large insurance portfolios, equity risk is the (by far) dominant risk source for the resulting liabilities (in the absence of a hedging portfolio).

Insurance liabilities are influenced by various sources of risk such as equity, interest, and insurance-specific risk. The interaction of these sources can be quite complex, so that the individual risk contributions are typically neither obvious nor readily available. This is particularly the case in life insurance, where the final payoffs – that commonly occur years or even decades after the origination of the contracts – depend on the interaction of financial factors and guarantees, aggregate demography trends, and actual deaths observed in the portfolio of insured. Nonetheless, insurance companies need to asses the relative importance of each source of risk in order to be able to devise adequate risk management strategies. This may simply be a matter of identifying the most significant source of risk for focusing efforts in case resources for risk management are limited (Hoem, 1988; Kling et al., 2014). Alternatively, the decomposition may allow to gage the sufficiency of risk loadings to each source of risk taking into account its contribution to the aggregate risk (Christiansen, 2013; Niemeyer, 2014). Evaluating the impact of different sources of risk is also important in view of product design, particularly when there are different risk penalties for different sources of risk (Kochanski and Karnarski, 2011), and in view of Solvency II, where individual risk contributions need to be quantified explicitly in partial internal models. In addition, the decomposition may also help to adequately calibrate the Solvency II standard formula.

In contrast to the majority of literature, we focus in this paper on decomposing the random variable "life insurance liabilities" into risk factors, which are again random variables isolated from the total risk and associated to one of the sources of risk, and not on quantifying the risk contributions themselves by means of risk measures. Each allocation principle for quantifying risk contributions, as e.g. Euler's allocation or the variance decomposition, is eventually based on a specific risk factor decomposition. As a result, only a comprehensive understanding of the underlying risk factor decomposition enables a reliable interpretation of the deduced risk contributions. In addition, risk factor decompositions also allow a detailed analysis of the influence of different sources of risk on a distributional level. For example, they are able to provide information about the dependency structure of different risk factors.

Given the relevance of risk decompositions, it is not surprising that there are a number of papers suggesting different methodologies for deriving risk factors, particularly in the life insurance context. Bühlmann (1995), Fischer (2004), Martin and Tasche (2007), and Christiansen and Helwich (2008) use

[1] For corresponding contributions, see Bühlmann (1995); Fischer (2004); Martin and Tasche (2007); Christiansen (2007); Christiansen and Helwich (2008); Artinger (2010); Rosen and Saunders (2010); Gatzert and Wesker (2014); Karabey et al. (2014), among others.

a conditional expectation approach, which is the probabilistic foundation of the well-known variance decomposition. Another approach also based on conditional expectations – the so-called Hoeffding decomposition – is used, for example, by Rosen and Saunders (2010). The Taylor expansion method (Christiansen, 2007) uses derivatives for decomposing functionals of different sources of risk. A completely different method, which is also implied by the Solvency II framework and used in Gatzert and Wesker (2014) and Artinger (2010), switches off the randomness of all sources of risk which are momentarily not under consideration. Karabey et al. (2014) rely on a number of these approaches (Hoeffding, Taylor, and conditional variance decomposition), and show how the contributions of different sources of risk can be derived from the risk decompositions using the Euler allocation principle.

In this paper, we commence by introducing a number of properties that define a meaningful risk decomposition for insurance liabilities.[2] In particular, we posit that the decomposition should consider the entire distribution of the company's risk (P1),[3] that it should be unique (P3) and independent of the ordering of the risks (P4), that the different risk factors can be clearly attributed to the different sources of risk (P2) and that they are comparable to one another (P5), and finally that the decomposition should aggregate to the (normalized) entire risk (P6). However, it turns out that when benchmarking the decomposition approaches proposed in literature with this list of desirable properties, for each method at least one of the properties fails to hold.

This leads us to propose our alternative MRT decomposition. We will show that this approach satisfies each property P1 to P6, and furthermore that the risk factor associated with *unsystematic mortality risk* vanishes as the portfolio size increases – whereas the *systematic* risk factors approach a non-zero limit. We provide explicit formulas for the decomposition, assuming a very general definition of the payoff of the insurance contract entailing discrete as well as continuous survival and death benefits, by relying on the Clark-Ocone formula (in the general case) and Itô's Lemma for diffusion processes (in the Markov case).

Our detailed numerical example relies on an affine specification of the interest and the mortality rates following Cox et al. (1985) and Dahl and Møller (2006), respectively, and a geometric Brownian motion for the underlying Variable Annuity account. Thus, we decompose the liabilities associated with a return-of-premium GMDB – which presents a very common product in insurance practice – into four sources of risk: equity risk, interest rate risk, systematic mortality risk, and unsystematic mortality risk. Our calculations show that for an unhedged exposure, equity risk is by far the most dominant risk, particularly when considering moderately sized insurance portfolios. More advanced examples for Guaranteed Annuity Options and Guaranteed Minimum Income Benefits within Variable Annuities that also consider the impact of hedging will be considered in a subsequent paper.

From a technical perspective, the derivation of our MRT decomposition is closely related to quadratic hedging approaches for life insurance liabilities under a martingale measure (Møller, 2001; Barbarin, 2008; Dahl and Møller, 2006; Dahl et al., 2008; Biagini et al., 2012, 2013; Biagini and Schreiber, 2013; Norberg, 2013), with the conceptual difference that we operate under the physical measure since we are interested in risk assessments. As a result of the latter, we do not assume that discounted price processes are martingales, and the risk which cannot be traded in the market is explicitly split up between the different (non-tradable) sources of risk. We rely on the analogy to quadratic hedging approaches in our derivations, but we also present some new results in this direction such as the decomposition of

[2]Fischer (2004) also provides a list of desirable properties for a reasonable decomposition method. However, he focuses on a decomposition of life insurance liabilities into financial risk and unsystematic mortality risk, where a number of these properties are trivial or irrelevant (e.g., because of independence of the sources of risk).

[3]Most previous papers in the actuarial literature dealing with risk decomposition methods primarily aim at allocating the total risk, which is quantified by certain risk measures, to different sources of risk. For an overview and a numerical comparison of some decomposition and subsequent allocation approaches we refer to Karabey (2012). In contrast, the approaches listed above all propose a decomposition into random variables so that this property (P1) is satisfied for all of them.

arbitrary insurance payoffs within our general setting or the integration with the Clark-Ocone formula from Malliavin calculus.

The remainder of this paper is organized as follows. Section 2 presents the properties that define a meaningful risk decomposition, and analyzes whether conventional approaches from literature satisfy these properties. Section 3 lays out the considered life insurance modeling framework and introduces our MRT decomposition within this framework. Properties and the calculation of the MRT decomposition are discussed in Section 4. Section 5 describes and analyzes our Variable Annuity example. Finally, Section 6 concludes.

# 2 Meaningful risk decompositions

## 2.1 Definition of meaningful risk decompositions

As outlined in the introduction, the primary concern of this paper are decompositions of a (life) *insurer's total risk* – which we suppose is given via the (normalized) *loss* random variable $L$, $\mathrm{E}[L] = 0$ – into different risk factors $i = 1, 2, \ldots, k$. More precisely, we assume $k$ sources of risk, where $Z_i = (Z_i(t))_{0\leq t\leq T^*}$ denotes the $i$-th source of risk and $Z = (Z_1, \ldots, Z_k)$. We assume that the loss variable $L$ is $\sigma(Z)$-measurable, and consider decomposition methodologies which assign each source of risk a risk factor. While a number of papers in the actuarial literature have proposed a variety of decomposition methods, thus far there has been no systematic assessment and comparison among these different approaches. In what follows, we introduce a number of properties we argue a *meaningful risk decomposition* should satisfy. Equalities between random variables are in the almost sure sense.

**P1** Randomness

This property posits that the individual risk factors are given by *random variables*, say $R_1, R_2, \ldots, R_k$, where clearly random variable $R_i$ corresponds to risk factor $i \in \{1, 2, \ldots, k\}$. The primary idea is that rather than decomposing certain risk measures or metrics, we are looking for decompositions of the *randomness* of $L$ yielding the complete risk structure of each factor. We introduce the relation "$\leftrightarrow$" and write $(L, Z_1, \ldots, Z_k) \leftrightarrow (R_1, R_2, \ldots, R_k)$ to indicate that the loss $L$ depending on $(Z_1, \ldots, Z_k)$ corresponds to the decomposition $(R_1, R_2, \ldots, R_k)$.

**P2** Attribution

To assure that $R_i$ represents the risk factor related to risk $i$, we require that whenever the loss $L$ is $\sigma(Z_i)$-measurable and $Z_i$ is independent of $(Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k)$, then all other risk factors should be zero, i.e. $R_j = 0$ for all $i \neq j$.

**P3** Uniqueness

We expect the decomposition methodology to be unique or, in other words, not subject to problem-specific choices. Formally, we demand that $(L, Z_1, \ldots, Z_k) \leftrightarrow (R_1, R_2, \ldots, R_k)$ and $(L, Z_1, \ldots, Z_k) \leftrightarrow (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_k)$ implies $R_i = \tilde{R}_i$, $i \in \{1, 2, \ldots, k\}$.

**P4** Order invariance

This property suggests that the definition of the decomposition should be invariant to the order of the risks $1, 2, \ldots, k$. Formally, consider a permutation $\pi : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ and assume $(L, Z_1, \ldots, Z_k) \leftrightarrow (R_1, R_2, \ldots, R_k)$. Then we posit:

$$(L, Z_{\pi(1)}, \ldots, Z_{\pi(k)}) \leftrightarrow (R_{\pi(1)}, R_{\pi(2)}, \ldots, R_{\pi(k)}).$$

**P5** Scale invariance

We want the risk factors to be quantitatively comparable, even if they are related to different loss variables. Thus, the risk factors need to be be invariant against a change of scale in the source of risk. Formally, let $(L, Z_1, \ldots, Z_k) \leftrightarrow (R_1, R_2, \ldots, R_k)$, and define $\tilde{Z}_i(t) := f_i(Z_i(t))$ for all $i = 1, \ldots, k,\ 0 \leq t \leq T^*$, where $f_i : \mathbb{R} \to \mathbb{R}$ is a Borel measurable, invertible function which represents the change of scale in risk $i$. If $(L, \tilde{Z}_1, \ldots, \tilde{Z}_k) \leftrightarrow (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_k)$, then we require that $R_i = \tilde{R}_i$ for all $i \in \{1, \ldots, k\}$. (Note that $L$ is also $\sigma(\tilde{Z})$-measurable, where $\tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_k)$.)

**P6** Aggregation

The risk factor decomposition should aggregate to the total risk faced by the company, i.e. we posit that for each loss $L$ and risks $Z$ with $(L, Z_1, \ldots, Z_k) \leftrightarrow (R_1, R_2, \ldots, R_k)$ there exists a function $A_{(L,Z)} : \mathbb{R}^k \to \mathbb{R}$ such that

$$L = A_{(L,Z)}(R_1, R_2, \ldots, R_k).$$

**P6**$^*$ Additive aggregation

A special case of P6 is an additive aggregation function, i.e. the case where $L$ is given as the sum of the individual risk factors:

$$L = \sum_{i=1}^{k} R_i.$$

An additive decomposition is desirable for multiple reasons. For instance, it allows for the natural interpretation that the risk factors *sum up* to the total risk. Moreover, for a decomposition into summands, it is straightforward to derive decompositions for homogeneous risk measures as within the well-known Euler allocation principle (Karabey et al., 2014).

Note that the relation $\leftrightarrow$ is even a function if P3 is satisfied. Furthermore, if P6 additionally holds and the function $A_{(L,Z)}$ does not depend on $L$ (as e.g. satisfied under P6$^*$), then $\leftrightarrow$ is an injective function in $L$ for fixed $Z$ since

$$(R_1, \ldots, R_k) = (\tilde{R}_1, \ldots, \tilde{R}_k) \Rightarrow L = A_Z(R_1, \ldots, R_k) = A_Z(\tilde{R}_1, \ldots, \tilde{R}_k) = \tilde{L}.$$

## 2.2 Are conventional approaches meaningful?

Due to the relevance of the decomposition problem in actuarial practice, it is no surprise that a number of decomposition methods have been proposed in the risk and insurance literature. For describing and discussing these approaches with regard to the meaningful risk decomposition properties from Section 2.1, we consider the time-0 present value $L_0$ of an insurer's future losses and, for simplicity, assume that this random variable is only influenced by two risk drivers $Z_1 = (Z_1(t))_{0 \leq t \leq T^*}$ and $Z_2 = (Z_2(t))_{0 \leq t \leq T^*}$. The insurer's risk is identified with $L := L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$. Our results with respect to the meaningful risk decomposition properties are summarized in Table 1. We find that each considered decomposition approach from literature has at least one undesirable property. If these methods are still applied, one has to make sure that the respective disadvantages do not falsify the results. This is why in Section 3 we propose the MRT decomposition, which actually satisfies all meaningful risk decomposition properties (see Section 4.1). For completeness, we have also added the MRT decomposition to Table 1.

|  | P1 | P2 | P3 | P4 | P5 | P6 | P6* |
|---|---|---|---|---|---|---|---|
| Variance decomposition | ✓ | ✓ | ✓ | × | ✓ | ✓ | ✓ |
| Hoeffding decomposition | ✓ | ✓ | ✓ | ✓ | ✓ | × | × |
| Taylor expansion | ✓ | ✓ | × | ✓ | × | × | × |
| Solvency II approach | ✓ | × | × | ✓ | ✓ | × | × |
| MRT decomposition | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |

Table 1: Summary of whether conventional decomposition approaches and the proposed MRT decomposition satisfy (✓) the properties P1-P6* or not (×).

**Variance decomposition**

A very common approach in actuarial literature (and the closely related field of credit analysis) for decomposing the insurer's risk $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ into risk factors is a conditional expectation approach, which is the probabilistic foundation of the well-known variance decomposition. Bühlmann (1995) and Fischer (2004) use this approach to decompose the profit/loss of a life insurer into a financial and a biometric part. Martin and Tasche (2007) determine the systematic and unsystematic risk from a credit portfolio by this method, and Christiansen and Helwich (2008) extend the approach to three sources of risk of a life insurance portfolio, namely unsystematic and systematic mortality risk as well as financial risk, which implicitly paves the way for a general definition of the variance decomposition approach.

The basic idea is that the conditional expectation $R_1 := \mathrm{E}^{\mathbb{P}}\left(L \middle| Z_1\right)$ captures the randomness of $L$ caused by $Z_1$. Since the remaining risk $R_2 := L - R_1 = L - \mathrm{E}^{\mathbb{P}}\left(L \middle| Z_1\right)$ must represent the randomness caused by $Z_2$, the decomposition reads for $L = L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ as

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \left[\mathrm{E}^{\mathbb{P}}\left(L_0 \middle| Z_1\right) - \mathrm{E}^{\mathbb{P}}(L_0)\right] + \left[L_0 - \mathrm{E}^{\mathbb{P}}\left(L_0 \middle| Z_1\right)\right] = R_1 + R_2, \tag{2.1}$$

where $R_1$ and $R_2$ represent the two risk factors. As a result of the orthogonality property of conditional expectations it turns out that

$$\mathrm{Var}(L) = \mathrm{Var}(R_1) + \mathrm{Var}(R_2),$$

that is, the correlation of $R_1$ and $R_2$ is zero. Commonly, the latter equation is referred to as variance decomposition. However, since in the context of risk quantification the variance is often not the risk measure of choice, we restrict the following considerations to the general decomposition (2.1) and refer to this as variance decomposition. Note that for an arbitrary loss $L$ the variance decomposition directly implies that $\mathrm{E}^{\mathbb{P}}(R_1) = \mathrm{E}^{\mathbb{P}}(L)$ and $\mathrm{E}^{\mathbb{P}}(R_2) = 0$. Of course, this asymmetry is irrelevant when considering the variance but potentially relevant when applying different risk measures. This emphasizes the necessity to first standardize the loss $L_0$ to mean zero, i.e. considering $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$, and then apply the decomposition approach, resulting in $\mathrm{E}^{\mathbb{P}}(R_1) = \mathrm{E}^{\mathbb{P}}(R_2) = 0$.

Obviously, the risk factors $R_1$ and $R_2$ are random variables (P1) and they add up to the total risk (P6*/P6). Since conditional expectations are almost surely defined, the uniqueness of the variance decomposition also holds in the almost sure sense (P3). To check the attribution property P2, assume that $Z_1$ and $Z_2$ are independent. If $L$ is $\sigma(Z_1)$-measurable, then $R_2 = L - \mathrm{E}^{\mathbb{P}}\left(L \middle| Z_1\right) = L - L = 0$. Conversely, if $L$ is $\sigma(Z_2)$-measurable, then $L$ is independent of $Z_1$ and thus $R_1 = \mathrm{E}^{\mathbb{P}}\left(L \middle| Z_1\right) = \mathrm{E}^{\mathbb{P}}(L)$. Therefore, P2 is satisfied since $L$ is standardized to mean zero. The variance decomposition is also scale invariant (P5). To show this, let $f_1$ and $f_2$ be two Borel measurable, invertible functions and define $\tilde{Z}_i(t) := f_i(Z_i(t)),\ i = 1, 2$. Since $f_1$ and $f_2$ are invertible, it follows that $\sigma(\tilde{Z}_i) = \sigma(Z_i)$. As a result, $\tilde{R}_1 = \mathrm{E}^{\mathbb{P}}(L| \tilde{Z}_1) = \mathrm{E}^{\mathbb{P}}(L| Z_1) = R_1$ and $\tilde{R}_2 = L - \mathrm{E}^{\mathbb{P}}(L| \tilde{Z}_1) = L - \mathrm{E}^{\mathbb{P}}(L| Z_1) = R_2$. In contrast, the order invariance property P4 is not satisfied which is illustrated by the following example.

**Example 2.1.** *Assume that $L_0 = Z_1(T)Z_2(T)$, where $Z_1$ and $Z_2$ are two independent processes e.g. representing the number of survivors and the fund value of a portfolio of unit-linked pure endowment policies. Then it follows by the variance decomposition with respect to $Z = (Z_1, Z_2)$ that*

$$\begin{aligned} L_0 - \mathrm{E}^{\mathbb{P}}(L_0) &= \left[\mathrm{E}^{\mathbb{P}}(L_0|\, Z_1) - \mathrm{E}^{\mathbb{P}}(L_0)\right] + \left[L_0 - \mathrm{E}^{\mathbb{P}}(L_0|\, Z_1)\right] \\ &= \underbrace{\mathrm{E}^{\mathbb{P}}(Z_2(T))\left[Z_1(T) - \mathrm{E}^{\mathbb{P}}(Z_1(T))\right]}_{=R_1} + \underbrace{Z_1(T)[Z_2(T) - \mathrm{E}^{\mathbb{P}}(Z_2(T))]}_{=R_2}. \end{aligned}$$

*In contrast, switching the order of $Z_1$ and $Z_2$, i.e. considering $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2) := (Z_2, Z_1)$, the variance decomposition approach yields*

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \underbrace{\mathrm{E}^{\mathbb{P}}(Z_1(T))\left[Z_2(T) - \mathrm{E}^{\mathbb{P}}(Z_2(T))\right]}_{=\tilde{R}_1} + \underbrace{Z_2(T)[Z_1(T) - \mathrm{E}^{\mathbb{P}}(Z_1(T))]}_{=\tilde{R}_2}.$$

Of course, in general the distributions of $R_1$ and $\tilde{R}_2$ from Example 2.1, which both measure the risk caused by $Z_1$, are different since $\mathrm{E}^{\mathbb{P}}(Z_2(T))$ is replaced by $Z_2(T)$ (analogously with $R_2$ and $\tilde{R}_1$), so that depending on the order of the sources of risk different risk factors follow. As a result, the variance decomposition is not order invariant (P4). In particular, if in the above example $Z_1(T)$ and $Z_2(T)$ were two standard Brownian motions with mean zero, the first decomposition would imply $R_1 = 0$ and $R_2 = Z_1(T)Z_2(T)$, whereas the second decomposition would yield $\tilde{R}_1 = 0$ and $\tilde{R}_2 = Z_1(T)Z_2(T)$. This means that either no risk ($R_1$) or the total risk ($\tilde{R}_2$) is attributed to $Z_1$. Furthermore, since in this specific example $L_0$ is symmetric in $Z_1$ and $Z_2$ and both risks have the same distribution, none of the two decompositions appear to be reasonable.

As already mentioned, although $Z_1$ and $Z_2$ might be correlated, $R_1$ and $R_2$ are uncorrelated. This means that the correlated risk must be allocated in an independent way, which can result in random effects as demonstrated in the next example.

**Example 2.2.** *Let $L_0 = B_1(T) + B_2(T)$ , where $B_1$ and $B_2$ are two one-dimensional Brownian motions with $dB_1(t)dB_2(t) = \rho dt,\ \rho \in (-1,1)\backslash\{0\}$. Since there exists a one-dimensional Brownian motion $B_3$ independent of $B_1$ such that $B_2(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_3(t)$ (Exercise 4.16 in Shreve, 2004, p. 200), the first risk factor of the variance decomposition with respect to $(B_1, B_2)$ can be calculated as*

$$R_1 = \mathrm{E}^{\mathbb{P}}(L_0|B_1) - \mathrm{E}^{\mathbb{P}}(L_0) = \mathrm{E}^{\mathbb{P}}\left((1+\rho)B_1(T) + \sqrt{1-\rho^2}B_3(T)\middle|\, B_1\right) = (1+\rho)B_1(T),$$

*which depends on the correlation parameter. Naturally, we would rather expect $R_1 = B_1(T)$.*

This illustrates that the variance decomposition approach is not able to appropriately deal with correlations.

### Hoeffding decomposition

Next we consider a decomposition approach which is based on the Hoeffding decomposition from statistics and which is, for example, used by Rosen and Saunders (2010) to determine the factor contributions to a portfolio's credit risk. For convenience, we again call this approach Hoeffding decomposition. Similar to the previous approach it relies on conditional expectations. If the insurer's liability is given by $L_0 = F(Z_1, Z_2)$, i.e. $L_0$ is a function of the stochastic processes $Z_1$ and $Z_2$, then the Hoeffding decomposition reads for $L = L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ as

$$\begin{aligned} L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = &\underbrace{\mathrm{E}^{\mathbb{P}}(L_0|\, Z_1) - \mathrm{E}^{\mathbb{P}}(L_0)}_{=:R_1} + \underbrace{\mathrm{E}^{\mathbb{P}}(L_0|\, Z_2) - \mathrm{E}^{\mathbb{P}}(L_0)}_{=:R_2} \\ &+ \underbrace{\mathrm{E}^{\mathbb{P}}(L_0|\, Z_1, Z_2) - \mathrm{E}^{\mathbb{P}}(L_0|\, Z_1) - \mathrm{E}^{\mathbb{P}}(L_0|\, Z_2) + \mathrm{E}^{\mathbb{P}}(L_0)}_{=:R_{1,2}}, \end{aligned}$$

where $R_1$ and $R_2$ are the risk factors caused by $Z_1$ and $Z_2$ in isolation, and $R_{1,2}$ denotes the insurer's risk which results from joint effects of the two different sources of risk. This already exhibits the decomposition's main drawback, namely that the total risk is not completely allocated to the individual sources of risk. If in Example 2.1 $Z_1(T)$ and $Z_2(T)$ had both mean zero, the Hoeffding approach would yield $R_1 = R_2 = 0$ and $R_{1,2} = L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$, i.e. the total risk results from joint effects, which does not give any insights on the influence of the different sources of risk. This example shows that the aggregation property P6 is in general not satisfied since for every function $A_{(L,Z)} : \mathbb{R}^2 \to \mathbb{R}$ we have $A_{(L,Z)}(R_1, R_2) = A_{(L,Z)}(0,0) \equiv const. \neq Z_1(T)Z_2(T)$. In particular, the sum of the risk factors is not equal to the total risk (P6*). Since Example 2.2 is still valid for the Hoeffding decomposition, this method also fails in dealing with correlations. However, properties P1 to P5 are satisfied. Clearly, the risk factors $R_1$ and $R_2$ are random variables (P1) and the Hoeffding decomposition is unique in the almost sure sense as a result of the uniqueness of the conditional expectations (P3). Furthermore, it can be easily seen that, contrary to the variance decomposition, this approach is order invariant (P4). The scale invariance follows by the same argument as with the variance decomposition (P5). Let $Z_1$ and $Z_2$ be two independent processes. If $L$ is (w.l.o.g.) $\sigma(Z_1)$-measurable, then $L$ and thus also $L_0$ are independent of $Z_2$, so that $R_2 = \mathrm{E}^{\mathbb{P}}\left(L_0 \middle| Z_2\right) - \mathrm{E}^{\mathbb{P}}\left(L_0\right) = \mathrm{E}^{\mathbb{P}}\left(L_0\right) - \mathrm{E}^{\mathbb{P}}\left(L_0\right) = 0$. Therefore, the attribution property P2 is also satisfied.

**Taylor expansion**

Another approach, which makes use of derivatives instead of conditional expectations, is proposed by Christiansen (2007, p. 80). He takes up the idea from uncertainty analysis which is to approximate functionals of random variables by their first order Taylor expansion and to interpret the summands as risk factors. If the insurer's loss $L_0$ equals a functional of the form $F(Z_1(T), Z_2(T))$, then this approach yields for $L = L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ that

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) \approx \left[F(Z_1, z_2) - \mathrm{E}^{\mathbb{P}}(L_0)\right] + \underbrace{\frac{\partial F}{\partial z_1}(z_1, z_2)(Z_1(T) - z_1)}_{=:R_1} + \underbrace{\frac{\partial F}{\partial z_2}(z_1, z_2)(Z_2(T) - z_2)}_{=:R_2},$$

where $(z_1, z_2)$ denotes the (deterministic) expansion point. By generalizing the definition of gradients, Christiansen (2007) even extends this approach to an infinite-dimensional setting such that the loss $L_0$ may also depend on the entire path of the stochastic processes $Z_1$ and $Z_2$. Clearly, the method's applicability is restricted, since the derivatives do not necessarily exist. Besides this, a first-order Taylor expansion and its summands can only approximate the risk $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ given the functional is non-linear. In addition, the approximation error at a certain point highly depends on the choice of the expansion point, i.e. the Taylor expansion is local. The latter two aspects are demonstrated by the following example.

**Example 2.3.** *Assume that $L_0 = Z_1(T)Z_2(T) =: F(Z_1(T), Z_2(T))$, where $Z_1$ and $Z_2$ are two arbitrary stochastic processes. Then the Taylor expansion with expansion point $(z_1, z_2)$ yields*

$$\begin{aligned} L_0 - \mathrm{E}^{\mathbb{P}}(L_0) &\approx \left[z_1 z_2 - \mathrm{E}^{\mathbb{P}}(L_0)\right] + \underbrace{z_2(Z_1(T) - z_1)}_{=R_1} + \underbrace{z_1(Z_2(T) - z_2)}_{=R_2} \\ &= L_0 - \mathrm{E}^{\mathbb{P}}(L_0) - (Z_1(T) - z_1)(Z_2(T) - z_2). \end{aligned}$$

*Obviously, the approximation error amounts to $-(Z_1(T) - z_1)(Z_2(T) - z_2)$, i.e. the more $Z_1(T)$ and $Z_2(T)$ deviate from $z_1$ and $z_2$, respectively, the higher is the approximation error.*

*Moreover, in the special case where $Z_1$ and $Z_2$ are two independent standard Brownian motions, choosing the expansion point $(z_1, z_2) = (\mathrm{E}^{\mathbb{P}}(Z_1(T)), \mathrm{E}^{\mathbb{P}}(Z_2(T))) = (0,0)$ results in $R_1 = R_2 = 0$, i.e. a risk is neither allocated to $Z_1$ nor to $Z_2$, although $L = L_0 = Z_1(T)Z_2(T)$.*

Obviously, the additive aggregation property P6* is generally not satisfied. This extends to the more general aggregation property P6 as can be seen from Example 2.3 with $Z_1$ and $Z_2$ being two independent standard Brownian motions: for every function $A_{(L,Z)} : \mathbb{R}^2 \to \mathbb{R}$ we have $A_{(L,Z)}(R_1, R_2) = A_{(L,Z)}(0, 0) \equiv const. \neq Z_1(T)Z_2(T)$. As a result of the dependence on the expansion point, the Taylor expansion approach is also not unique (P3). For example, choosing the expansion point $(z_1, z_2) = (1, 0)$ in the second part of Example 2.3 would change the second risk factor to $R_2 = Z_2(T) \neq 0$. To show that scale invariance (P5) is violated, simply assume that $L = e^{Z_1(T)}$. Then the Taylor expansion yields $L \approx e^{z_1} + e^{z_1}(Z_1(T) - z_1)$ for some expansion point $z_1$. However, for $\tilde{Z}_1(T) := e^{Z_1(T)}$ and $\tilde{z}_1 := e^{z_1}$ we would have $L \approx \tilde{z}_1 + (\tilde{Z}_1(T) - \tilde{z}_1)$, and in general $R_1 = e^{z_1}(Z_1(T) - z_1) \neq e^{Z_1(T)} - e^{z_1} = \tilde{Z}_1(T) - \tilde{z}_1 = \tilde{R}_1$. Still, the Taylor expansion satisfies properties P1, P2, and P4. The risk factors are obviously random variables, and the order invariance can be easily shown. For the attribution property, let $Z_1$ and $Z_2$ be two independent processes, and assume that $L = L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ is (w.l.o.g.) $\sigma(Z_1)$-measurable, where $L_0 = F(Z_1(T), Z_2(T))$. Consequently, either $L_0 = F(Z_1(T), Z_2(T)) = F(Z_1(T))$, or $Z_2(T)$ is $\sigma(Z_1)$-measurable. In the first case, it follows that $\frac{\partial F}{\partial z_2}(z_1, z_2) = \frac{\partial F}{\partial z_2}(z_1) \equiv 0$ and thus $R_2 = 0$. The second case implies that $Z_2(T)$ must be deterministic, since $Z_1$ and $Z_2$ are independent. Since a natural choice in this case would be $z_2 = Z_2(T)$, this also yields $R_2 = \frac{\partial F}{\partial z_2}(z_1, z_2)(Z_2(T) - z_2) \equiv 0$. Both cases confirm the attribution property P2.

### Solvency II approach

A different risk decomposition approach is to switch off the randomness of all the sources of risk that are momentarily not under consideration, see e.g. Gatzert and Wesker (2014), Artinger (2010). Since this method is in principle also proposed by the Solvency II framework for measuring the influence of different sources of risk (cf. CEIOPS, 2010), in what follows we call this decomposition method Solvency II approach.

In more detail, given the functional $F(Z_1, Z_2)$ represents the insurer's risk, the method suggests to model the risk factors implied by $Z_1$ and $Z_2$ by $F(Z_1, z_2)$ and $F(z_1, Z_2)$, respectively. In the context of Solvency II, $z_1$ and $z_2$ are typically chosen as best estimates of $Z_1$ and $Z_2$. However, in general there is no clear answer how $z_1$ and $z_2$ should be chosen. In fact, the decomposition heavily depends on the choice of $z_1$ and $z_2$ and is thus not unique (P3). This is illustrated in the following example. In the interest of clarity, we do not standardize the risk to mean zero here.

**Example 2.4.** *Assume that* $L = F(Z_1(T), Z_2(T)) = Z_1(T) \max\{K - Z_2(T), 0\}$, *where* $Z_1$ *and* $Z_2$ *are two arbitrary processes and* $K$ *a constant. For example, imagine the premium maintenance guarantee of a portfolio of unit-linked pure endowment policies, where* $Z_1$ *describes the number of survivors,* $Z_2$ *the fund value and* $K$ *the single premium. It is natural to assume that* $\mathrm{E}^{\mathbb{P}}(Z_2(T)) > K$, *but that* $\mathbb{P}(Z_2(T) < K) > 0$. *Measuring the risk factor related to* $Z_1$ *by replacing* $Z_2(T)$ *with its expectation, the Solvency II approach yields*

$$R_1 = F(Z_1(T), \mathrm{E}^{\mathbb{P}}(Z_2(T))) = Z_1(T) \max\{K - \mathrm{E}^{\mathbb{P}}(Z_2(T)), 0\} \equiv 0.$$

*Although* $L_0 > 0$ *with positive probability (when additionally assuming that some policyholders survive until time* $T$ *with positive probability), no risk is allocated to* $Z_1$. *However, choosing any deterministic approximation* $z_2(T) < K$ *would yield* $R_1 = Z_1(T)(K - z_2(T))$ *with a different distribution for each choice of* $z_2(T)$. *Assuming* $z_1(T) = \mathrm{E}^{\mathbb{P}}(Z_1(T))$, *the second risk factor equals here* $R_2 = \mathrm{E}^{\mathbb{P}}(Z_1(T)) \max\{K - Z_2(T), 0\}$.

Besides uniqueness, the example implies that the Solvency II approach also does not satisfy the aggregation property P6 (and thus also not P6*). In more detail, let $Z_1$ and $Z_2$ be independent and assume that there exists a function $A_{(L,Z)} : \mathbb{R}^2 \to \mathbb{R}$ such that $L = A_{(L,Z)}(R_1, R_2) = A_{(L,Z)}(0, \mathrm{E}^{\mathbb{P}}(Z_1(T)) \max\{K-$

$Z_2(T), 0\})$. This would imply that $L$ is $\sigma(Z_2)$-measurable and thus independent of $Z_1$ which is obviously a contradiction (assuming that $Z_1$ is non-deterministic). The attribution property also does not hold (P2). To see this, consider two sources of risk $Z_1$ and $Z_2$ and assume that $L = Z_1(T) = F(Z_1(T), Z_2(T))$. Obviously, $L$ is $\sigma(Z_1)$-measurable, but for every $z_1(T) \neq 0$ it holds that $R_2 = F(z_1, Z_2) = z_1(T) \neq 0$. The only properties that are satisfied by the Solvency II approach are P1, P4, and P5. Again, the risk factors are obviously random variables (P1) and the order invariance can be easily shown (P4). For the scale invariance, let $f_1$ and $f_2$ be two Borel measurable, invertible functions and define $\tilde{Z}_i(t) := f_i(Z_i(t))$ and $\tilde{z}_i(t) := f_i(z_i(t)),\ i = 1, 2$. It follows that $L = F(Z_1, Z_2) = F((f_1^{-1}(\tilde{Z}_1(t)))_{0\le t\le T^*}, (f_2^{-1}(\tilde{Z}_2(t)))_{0\le t\le T^*}) =: \tilde{F}(\tilde{Z}_1, \tilde{Z}_2)$. As a result, $\tilde{R}_1 = \tilde{F}(\tilde{Z}_1, \tilde{z}_2) = F(Z_1, z_2) = R_1$ and analogously $\tilde{R}_2 = R_2$, which proves the scale invariance (at least if the change of scale is the same for $\tilde{z}_i$ as for $\tilde{Z}_i,\ i = 1, 2$) (P5).

Generally, whenever the probability of an option to be in the money is rather low, the deterministic approximation might be misleading. Furthermore, there is no obvious reason why not the considered source of risk should be switched off, then measuring the risk factors caused by $Z_1$ and $Z_2$ via $L - F(z_1, Z_2)$ and $L - F(Z_1, z_2)$ instead of via $F(Z_1, z_1)$ and $F(z_1, Z_2)$, respectively.

# 3 MRT decomposition in life insurance

## 3.1 Life insurance modeling framework

For the remainder of this paper, we fix a finite time horizon $T^*$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T^*}$ satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness.[4] Throughout, $\mathcal{F}_t$ describes the total information available at time $t$, in particular let $\mathcal{F}_0$ be trivial and set $\mathcal{F} = \mathcal{F}_{T^*}$. We assume that the entire uncertainty of the life insurer's future profit/loss arises from the uncertain evolution of a number of financial and demographic factors as well as the actual occurrence of deaths in the insurance portfolio. We introduce an $n$-dimensional locally bounded process $X = ((X_1(t), \dots, X_n(t))^\top)_{0\le t\le T^*}$, the so-called *state process*, and assume that all financial and demographic factors are functions of $X$. Specifically, we assume that the time-$t$ prices of all risky assets from the financial market, as well as the short rate $r(t) = r(t, X(t))$ and the mortality intensity $\mu(t) = \mu(t, X(t))$ can be expressed in terms of $X(t)$. The state process itself is driven by a $d$-dimensional standard Brownian motion $W = ((W_1(t), \dots, W_d(t))^\top)_{0\le t\le T^*}$. Let $\mathbb{G}$ denote the augmented filtration generated by $W$ which is assumed to be a sub-filtration of $\mathbb{F}$. Furthermore, we impose the existence of a bank account $(B(t))_{0\le t\le T^*}$ defined as $B(t) = e^{\int_0^t r(s)ds}$. The financial market is assumed to be frictionless and arbitrage-free, the latter in the sense that there exists a risk-neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which payment streams can be evaluated via the expectation of their discounted values (with respect to the numéraire $B$).

For simplicity, we assume that the considered $m$ policyholders are all homogeneous and of age $x$ at time $0$. The remaining lifetime $\tau_x^i$ of the $i$-th policyholder as seen from time $0,\ i = 1, \dots, m$, is defined as the first jump time of a doubly stochastic process with $\mathbb{G}$-predictable intensity $(\mu(t))_{0\le t\le T^*}$, i.e.

$$\tau_x^i = \inf\left\{t \in [0, T^*] : \int_0^t \mu(s)ds \ge E_i\right\},\ \ i = 1, \dots, m,$$

where $E_i,\ i = 1, \dots, m$, are i.i.d. unit exponential random variables independent of $\mathcal{G}_{T^*}$. We use the convention $\inf \emptyset = \infty$, which covers the case $\tau_x^i > T^*$. A motivation for this definition of the remaining

[4] In principle, $\mathbb{P}$ can be any probability measure since all statements in this paper are generally valid. Still, in what follows we interpret $\mathbb{P}$ as the real-word measure, since we focus on risk.

lifetimes can be found in Biffis et al. (2010, p. 287). Assuming that $\mu$ is non-negative and continuous it follows from the definition that for any $t \in [0, T^*]$ (Lando, 1998, p. 102)

$$\mathbb{P}\left(\tau_x^i > T | \mathcal{G}_T\right) = e^{-\int_0^T \mu(s)ds},$$

and in particular $\mathbb{P}(\tau_x^i > 0) = 1$. The construction of the $\tau_x^i$'s also implies that (Bielecki and Rutkowski, 2004, p. 268)

$$\mathbb{P}\left(\tau_x^i > t \middle| \mathcal{G}_T\right) = \mathbb{P}\left(\tau_x^i > t \middle| \mathcal{G}_s\right) \tag{3.1}$$

for all $0 \leq t \leq s \leq T \leq T^*$, $i = 1, \ldots, m$. According to Jeanblanc and Rutkowski (2000), this is equivalent to the so-called $\mathcal{H}$-hypothesis, which says that every $\mathbb{G}$-martingale remains a martingale with respect to the larger filtration $\mathbb{F}$. The process $\Gamma(t) := \int_0^t \mu(s)ds$ is called cumulative mortality intensity of the random times $\tau_x^i$, $i = 1, \ldots, m$. It follows that $\mathbb{P}\left(\tau_x^i > t \middle| \mathcal{G}_t\right) = e^{-\Gamma(t)}$. Summing up, henceforth we assume that $(\mu(t))_{0 \leq t \leq T^*}$ is continuous, non-negative and $\mathbb{G}$-predictable.

Defining the sub-filtration $\mathbb{I} = \bigvee_{i=1}^m \mathbb{I}^i$ of $\mathbb{F}$, where $\mathbb{I}^i = (\mathcal{I}_t^i)_{0 \leq t \leq T^*}$ is the augmented filtration generated by the death indicator process $(\mathbb{1}_{\{\tau_x^i \leq t\}})_{0 \leq t \leq T^*}$, it is natural to assume that $\mathbb{F}$ is given by $\mathbb{G} \vee \mathbb{I}$ ($\mathbb{G}$ as defined above). This means that we distinguish between the world of financial and demographic factors on the one hand, and the occurrence of deaths on the other hand.

The above definitions imply that the residual lifetimes $\tau_x^i$, $i = 1, \ldots, m$, of the homogeneous policyholders are conditionally identically distributed and conditionally independent given the $\sigma$-algebra $\mathcal{G}_{T^*}$. Since $\{\tau_x^i > t\}$ is an atom of $\mathcal{I}_t^i$, the construction of $\mathbb{F}$ together with the conditional independence property imply for the conditional survival probability given $\mathcal{F}_t$ that (Bielecki and Rutkowski, 2004, p. 145)

$$\mathbb{P}\left(\tau_x^i > T | \mathcal{F}_t\right) = \mathbb{P}\left(\tau_x^i > T | \mathcal{G}_t \vee \mathcal{I}_t^i\right) = \mathbb{1}_{\{\tau_x^i > t\}} \frac{\mathbb{P}\left(\tau_x^i > T | \mathcal{G}_t\right)}{\mathbb{P}\left(\tau_x^i > t | \mathcal{G}_t\right)} = \mathbb{1}_{\{\tau_x^i > t\}} \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T \mu(s)ds} \middle| \mathcal{G}_t\right).$$

Each life insurance contract from the considered portfolio is assumed to entail the same cash flows. The only difference is the respective remaining lifetime. We describe the number of policyholders who have died until time $t$ by $N(t) = \sum_{i=1}^m \mathbb{1}_{\{\tau_x^i \leq t\}}$. Then the sum of the (possibly discounted) future cash flows as from time $t$, which represents the *insurer's total net liability*, is given by

$$\begin{aligned} L_t = \; & C_t + \sum_{k:\; t_k \geq t} (m - N(t_k)) C_{a,k} + \int_t^{T^*} (m - N(s)) C_a(s) ds \\ & + \sum_{k:\; t_k \geq t} (N(t_k) - N(t_{k-1})) C_{ad,k} + \int_t^{T^*} C_{ad}(s) dN(s). \end{aligned} \tag{3.2}$$

Clearly, cash flows related to a life insurance contract can depend on the policyholder's survival or his death, but they can also be independent of the individual lives as e.g. cash flows from hedging or benefit payments from a fixed term insurance. Accordingly, the first term in equation (3.2) summarizes all cash flows which are independent of the individual lives, the second and the third term describe cash flows which are conditional on survival, and the last two terms describe cash flows which are conditional on death.

While in practice cash flows are usually only generated at discrete points in time (e.g. daily or monthly), in literature also continuously generated cash flows are considered (e.g., Dahl et al., 2008, p. 125). This is why we consider in the second and in the fourth term discrete cash flows which are triggered by survival or being death at certain discrete points in time $0 = t_0 < t_1 < \ldots < t_l = T^*$, $l \in \mathbb{N}$.

In the third and in the fifth term we describe continuous payments which are instantly triggered by survival and death. The detailed description of the cash flows used above is as follows:

- $C_t$ sum of all (possibly discounted) payments at or after time $t$ which are independent of $\tau_x^i,\ i = 1, \ldots, m,\ t \in [0, T^*]$, e.g. hedging returns, benefits from a fixed term insurance;
- $C_{a,k}$ sum of all (possibly discounted) payments at or after time $t_k$ which are conditional on survival until time $t_k,\ k = 0, \ldots, l$, e.g. single premiums, discrete premium payments, discrete annuity payments, benefits from a pure endowment, benefits within the period certain of a deferred annuity;
- $C_a(t)$ time-$t$ intensity of all continuous payments which are conditional on survival until time $t$, or in other words, $C_a(t)dt$ is the sum of all payments in the infinitesimal period $[t, t + dt]$ which are conditional on survival until time $t$, e.g. continuous premium payments, continuous annuity payments;
- $C_{ad,k}$ sum of all (possibly discounted) payments at or after time $t_k$ which are conditional on death within $(t_{k-1}, t_k],\ k = 1, \ldots, l$, e.g. death benefits paid at the end of a period;
- $C_{ad}(t)$ sum of all (possibly discounted) payments at time $t$ which are conditional on death at time $t$, $t \in [0, T^*]$, e.g. death benefits paid immediately after death.

We assume that $C_t, C_{a,k}$ and $C_{ad,k}$ are $\mathcal{G}_{T^*}$-measurable, which means that the cash flows may only be known at time $T^*$, whereas $C_a(t)$ and $C_{ad}(t)$ are assumed to be $\mathcal{G}_t$-measurable. The latter assumption is one of the reasons why we explicitly distinguish between discrete and continuous cash flows. In general, each cash flow from above may include several payments from and to the insurance company. Positive payments are interpreted as payments made by the insurer, and negative payments are interpreted as payments received by the insurance company. Thus, each cash flow corresponds to the insurer's net liability which justifies the interpretation of $L_t$ as the insurer's total net liability.

The *insurer's risk* at time $t$ (as seen from time 0) is identified with $L_t - \mathrm{E}^{\mathbb{P}}\left( L_t \middle| \mathcal{F}_t \right)$, i.e. the insurer's net liability as from time $t$ less the liability's expectation given the development until time $t$. The net liability $L_t$ is exactly the amount of money the insurance company needs at time $t$ in order to be able to meet its future contract obligations (under certain investment assumptions introduced by the discount factor). Of course, this amount of money is random and the insurance company should at least bargain for the expected value of $L_t$ which is often the basis for the insurer's reserve. Thus, subtracting $\mathrm{E}^{\mathbb{P}}\left( L_t \middle| \mathcal{F}_t \right)$ yields the risk the insurer actually has to face.

Note that similar frameworks as above have already been considered by, among others, Bauer et al. (2010) and Zhu and Bauer (2011).

## 3.2 Definition of the MRT decomposition

Within the life insurance modeling framework introduced in the previous section, the objective is to find an approach which decomposes the insurer's risk $L_t - \mathrm{E}^{\mathbb{P}}\left( L_t | \mathcal{F}_t \right)$ in such a way that the meaningful risk decomposition properties formulated in Section 2.1 are satisfied. Inspired by the martingale representation theorem, we propose to decompose the insurer's risk into stochastic integrals with respect to the compensated sources of risk and to interpret each integral as the risk factor of the respective source of risk. As we will see in Section 4.1, this approach which is further specified below satisfies all requirements established in Section 2.1.

The sources of risk are identified, on the one hand, with the state processes $X_i = (X_i(t))_{0 \le t \le T^*},\ i =$

$1, \ldots, n$, and on the other hand with the number of deaths in the portfolio $N = (N(t))_{0 \le t \le T^*}$ as introduced in Section 3.1. The respective compensated processes, i.e. the processes less their $\mathbb{F}$-compensators, are denoted by $M_i^W = (M_i^W(t))_{0 \le t \le T^*}$, $i = 1, \ldots, n$, and $M^N = (M^N(t))_{0 \le t \le T^*}$, respectively. In what follows, all processes are assumed to be semimartingales, so that, in particular,the stochastic integrals below are well-defined.

For the sake of clarity, we specify the approach for the case $t = 0$, but all results can be generalized to any time $t$. Accordingly, we look for a decomposition

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \sum_{i=1}^{n} \int_0^{T^*} \psi_i^W(t) dM_i^W(t) + \int_0^{T^*} \psi^N(t) dM^N(t), \tag{3.3}$$

where $\psi_i^W(t),\ i = 1 \ldots, n$, and $\psi^N(t)$ are $\mathbb{F}$-predictable processes. A decomposition of the form (3.3) is henceforth called MRT decomposition, since the idea and the decomposition's existence (see Proposition 3.3) are implied by the martingale representation theorem. Each integral is interpreted as the portion of the total randomness of $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ which is caused by the associated source of risk. In particular, $\int_0^{T^*} \psi^N(t) dM^N(t)$ describes the randomness introduced by $N$, i.e. by the random occurrence of deaths in the portfolio, and thus corresponds to the inherent unsystematic mortality risk. Note that a decomposition consisting of stochastic integrals with respect to the different sources of risk $X_i,\ i = 1, \ldots, n$, and $N$ (instead of the compensated processes) does not necessarily exist, since most risk processes are not $\mathbb{P}$-martingales. Just imagine that $X(t) := t + W(t)$ and assume that we could find for every $L_0 := W(t),\ 0 \le t \le T^*$, an accordant decomposition with respect to the source of risk $X = (X(t))_{0 \le t \le T^*}$. Since $W$ is a martingale, this would yield a contradiction . In contrast, we will show in Proposition 3.3 that a decomposition of the form (3.3) always exists given certain natural conditions are satisfied.

For simplicity, we focus on the special case of Itô processes as state processes.

**Assumption 3.1.** *The state process $X = ((X_1(t), \ldots, X_n(t))^\top)_{0 \le t \le T^*}$ is an $n$-dimensional Itô process satisfying*

$$dX(t) = \theta(t)dt + \sigma(t)dW(t), \tag{3.4}$$

*with deterministic initial value $X(0) = x_0 \in \mathbb{R}^n$, where the $n$-dimensional drift vector $\theta = (\theta(t))_{0 \le t \le T^*}$ and the $n \times d$-dimensional volatility matrix $\sigma = (\sigma(t))_{0 \le t \le T^*}$ are both $\mathbb{G}$-adapted with continuous paths. We assume that there exists a unique strong solution to* (3.4).

With the previous assumption, we can determine the compensators of the sources of risk. A proof can be found in Appendix A.

**Lemma 3.2.** *Under Assumption 3.1 we have that:*

*i) The unique compensator of $X_i$ is given by $A_i^W = (A_i^W(t))_{0 \le t \le T^*}$, where $A_i^W(t) = \int_0^t \theta_i(s)ds,\ i = 1, \ldots, n$. Thus,*

$$M_i^W(t) = \sum_{j=1}^{d} \int_0^t \sigma_{ij}(s) dW_j(s), \quad 0 \le t \le T^*,\ i = 1, \ldots, n.$$

*ii) The unique compensator of $N$ is given by $A^N = (A^N(t))_{0 \le t \le T^*}$, where $A^N(t) = \int_0^t (m - N(s-))\mu(s)ds$. Thus,*

$$M^N(t) = N(t) - \int_0^t (m - N(s-))\mu(s)ds, \quad 0 \le t \le T^*.$$

Next we show that in the case $n = d$ a decomposition of the form (3.3) exists and is unique.

**Proposition 3.3.** *Let the state process $X$ be defined as in Assumption 3.1. If $n = d$, $\det \sigma(t) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, and $L_0$ is integrable, then there exist $\mathbb{F}$-predictable processes $\psi_1^W, \ldots, \psi_n^W, \psi^N : [0, T^*] \times \Omega \to \mathbb{R}$ such that the MRT decomposition (3.3) holds. The representation is unique in the sense that the integrands $\psi_1^W, \ldots, \psi_n^W, \psi^N$ are almost surely unique on $[0, T^*] \times \Omega$ with respect to $\lambda \otimes \mathbb{P}$, where $\lambda$ denotes the Lebesgue measure on $[0, T^*]$. Moreover, if $L_0$ is square integrable, then*

$$\mathrm{E}^{\mathbb{P}} \left( \left[ \int_0^{T^*} \psi^N(t) dM^N(t) \right]^2 \right) < \infty. \tag{3.5}$$

*Proof.* Note that $L_0$ is $\mathcal{F}_{T^*}$-measurable since all cash flows are assumed to be $\mathcal{G}_{T^*}$-measurable and $N(t)$ is by construction $\mathcal{F}_{T^*}$-measurable for all $t \in [0, T^*]$. By the martingale representation theorem in a Lévy setting (Jeanblanc et al., 2009, p. 621) applied to the martingale

$$M(t) := \mathrm{E}^{\mathbb{P}} \left( L_0 - \mathrm{E}^{\mathbb{P}}(L_0) \middle| \mathcal{F}_t \right), \ 0 \leq t \leq T^*,$$

together with the $\mathcal{F}_{T^*}$-measurability of $L_0$, it follows that there exist unique predictable processes $\widetilde{\psi}_1^W, \ldots, \widetilde{\psi}_d^W, \psi^N : [0, T^*] \times \Omega \to \mathbb{R}$ such that

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \int_0^{T^*} \widetilde{\psi}^W(t) dW(t) + \int_0^{T^*} \psi^N(t) dM^N(t), \tag{3.6}$$

where $\widetilde{\psi}^W := (\widetilde{\psi}_1^W, \ldots, \widetilde{\psi}_d^W)$. Since $n = d$ and $\det \sigma(t) \neq 0$ by assumption, the inverse of $\sigma$ exists (and is unique). Thus, if $\psi_i^W(t) := \sum_{j=1}^d \widetilde{\psi}_j^W(t) \sigma_{ji}^{-1}(t), \ i = 1, \ldots, n,$ denotes the $i$-th entry of the vector $\widetilde{\psi}^W(t)\sigma^{-1}(t)$, the first summand of (3.6) can be transformed into

$$\int_0^{T^*} \widetilde{\psi}^W(t) dW(t) = \int_0^{T^*} \widetilde{\psi}^W(t) \sigma^{-1}(t) \sigma(t) dW(t) = \sum_{i=1}^n \int_0^{T^*} \psi_i^W(t) dM_i^W(t),$$

which together with (3.6) proves the existence of the MRT decomposition (3.3).

The uniqueness of the MRT decomposition is a result of the uniqueness of $\widetilde{\psi}_1^W, \ldots, \widetilde{\psi}_n^W, \psi^N$, and the uniqueness of the inverse of $\sigma$. Moreover, if $L_0$ is square integrable, it follows that the martingale $M$ defined above is also square integrable, and Proposition 11.2.8.1 in Jeanblanc et al. (2009, p. 621) directly yields (3.5).

□

If $n \neq d$, existence and uniqueness of the MRT decomposition (3.3) are not necessarily given. In fact, as follows from the proof, we need to look for $\psi^W(t)$ such that the equation $\widetilde{\psi}^W(t) = \psi^W(t)\sigma(t)$ holds true, where $\widetilde{\psi}^W(t)$ results from the martingale representation theorem. If $n > d$, there are less equations than unknowns so that uniqueness is not guaranteed. On the other hand, if $n < d$, there are more equations than unknowns so that the existence might be violated. In what follows, we focus on the case $n = d$. If $n \neq d$, we assume that either redundant state processes (which can be represented via other state processes) are removed or additional state processes are artificially added, both along with an adjustment of the interpretation of the risk factors. In contrast to a hedging problem, where the number of state processes – or rather securities – is exogenously given, this procedure seems to be reasonable for a risk decomposition problem.

# 4 Analysis of the MRT decomposition

## 4.1 Meaningful risk decomposition properties

We now come back to the properties from Section 2.1 that we posit a meaningful risk decomposition method should satisfy. Recall that the MRT decomposition defined in Section 3.2 reads as

$$\underbrace{L_0 - \mathrm{E}^{\mathbb{P}}(L_0)}_{=:L} = \sum_{i=1}^{n} \underbrace{\int_0^{T^*} \psi_i^W(t)dM_i^W(t)}_{=:R_i} + \underbrace{\int_0^{T*} \psi^N(t)dM^N(t)}_{=:R_{n+1}}.$$

**Proposition 4.1.** *Assume that the state process $X = (X_1, \ldots, X_n)$ is defined as in Assumption 3.1 with $n = d$ and $\det \sigma(t) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Furthermore, let $L_0$ be integrable. Then the MRT decomposition $(L, X_1, \ldots, X_n, N) \overset{MRT}{\leftrightarrow} (R_1, \ldots, R_{n+1})$ as defined in* (3.3) *satisfies the properties P1, P2, P3, P4 , P6, and P6*. If $f_i : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable for each $i = 1, \ldots n$, and if $(f_{n+1}(N(t)))_{0 \leq t \leq T^*}$ is a again a counting process as $N$ defined in Section 3.1, then P5 is satisfied as well.*

*Proof.* Obviously, the risk factors $R_1, \ldots, R_{n+1}$ are random variables, and $L = \sum_{i=1}^{n+1} R_i$, so that P1 and P6* (and thus also P6) are satisfied. The uniqueness property P3 directly follows from Proposition 3.3. To simplify the proof of the remaining properties, we define $\psi_i := \psi_i^W,\ M_i := M_i^W,\ i = 1, \ldots, n$, and $\psi_{n+1} := \psi^N,\ M_{n+1} := M^N$. Furthermore, define $Z = (Z_1, \ldots, Z_{n+1}) := (X_1, \ldots, X_n, N)$. Assume that $(L, Z_1, \ldots, Z_{n+1}) \overset{MRT}{\leftrightarrow} (R_1, \ldots, R_{n+1})$.

P2: Let $i \in \{1, \ldots, n+1\}$ be arbitrary, but fixed. Assume that $L$ is $\sigma(Z_i)$-measurable and that $Z_i$ is independent of $Z_{i-} := (Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_{n+1})$. This implies that $L$ is independent of $Z_{i-}$ as well. Define

$$L(t) := \sum_{i=1}^{n+1} \int_0^t \psi_i(s)dM_i(s) = \mathrm{E}^{\mathbb{P}}\left(L|\, \mathcal{F}_t\right),$$

where the latter equation follows from the martingale property of the considered integrals. Since we assume that $\det \sigma(t) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, it follows that

$$\mathcal{F}_t = \mathcal{F}_t^Z = \mathcal{F}_t^{Z_i} \vee \mathcal{F}_t^{Z_{i-}},$$

where $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{0 \leq t \leq T^*}$ denotes the augmented filtration generated by $Z$, and accordingly $\mathbb{F}^{Z_i} = (\mathcal{F}_t^{Z_i})_{0 \leq t \leq T^*}$ and $\mathbb{F}^{Z_{i-}} = (\mathcal{F}_t^{Z_{i-}})_{0 \leq t \leq T^*}$. Since $\mathcal{F}_t^{Z_{i-}}$ is independent of $L$ and of $\mathcal{F}_t^{Z_i}$, we conclude that $\mathrm{E}^{\mathbb{P}}\left(L|\, \mathcal{F}_t\right) = \mathrm{E}^{\mathbb{P}}\left(L|\, \mathcal{F}_t^{Z_i}\right)$. As a result, the process $(L(t))_{0 \leq t \leq T^*}$ is independent of each process $Z_j,\ j \neq i$, and thus the predictable covariation process satisfies $\langle L, Z_j \rangle(t) = 0$ for all $j \neq i,\ 0 \leq t \leq T^*$.

(a) Assume that $i = n+1$. Then $\langle M_i, Z_j \rangle(t) = \left\langle M^N, X_j \right\rangle(t) = 0$ for all $j \neq i$, so that

$$0 = d\langle L, Z_j \rangle(t) = \sum_{k=1}^{n+1} \psi_k(t)d\langle M_k, Z_j \rangle(t) = \sum_{k=1}^{n} \psi_k(t)d\langle M_k, Z_j \rangle(t)$$
$$= \sum_{k=1}^{n} \psi_k(t)\sigma_{k,\cdot}(t)\sigma_{j,\cdot}^\top(t)dt,\ \ j \neq i,\ 0 \leq t \leq T^*,$$

where $\sigma_{k,\cdot}(t)$ denotes the $k$-th row of $\sigma(t)$. Thus, for any $0 \leq t \leq T^*$, this yields the linear system of equations $A_t^\top \psi_t = 0$, where

$$A_t := \begin{pmatrix} \sigma_{1,\cdot}(t)\sigma_{1,\cdot}(t)^\top & \dots & \sigma_{1,\cdot}(t)\sigma_{n,\cdot}(t)^\top \\ \vdots & & \vdots \\ \sigma_{n,\cdot}(t)\sigma_{1,\cdot}(t)^\top & \dots & \sigma_{n,\cdot}(t)\sigma_{n,\cdot}(t)^\top \end{pmatrix}, \ \psi_t := \begin{pmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{pmatrix}.$$

However, $A_t = \sigma(t)\sigma(t)^\top$ so that

$$\det A_t^\top = \det A_t = \det \sigma(t)\sigma(t)^\top = (\det \sigma(t))^2 \neq 0$$

for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely which implies $\psi_t = 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Thus, we have $R_j = \int_0^{T^*} \psi_j(t) dM_j(t) = 0$ almost surely for all $j \neq i$.

(b) Now assume that $i \neq n+1$ (w.l.o.g. $i = 1$). Then we know that

$$\begin{aligned} 0 &= d\langle L, Z_{n+1}\rangle(t) = \sum_{k=1}^{n+1} \psi_k(t) d\langle M_k, Z_{n+1}\rangle(t) = \psi_{n+1}(t) d\langle M_{n+1}, Z_{n+1}\rangle(t) \\ &= \psi_{n+1}(t) d\langle M_{n+1}, M_{n+1}\rangle(t). \end{aligned}$$

By the Itô isometry it follows that

$$\mathrm{E}^{\mathbb{P}}\left(\left[\int_0^{T^*} \psi_{n+1}(t) dM_{n+1}(t)\right]^2\right) = \mathrm{E}^{\mathbb{P}}\left(\int_0^{T^*} \psi_{n+1}^2(t) d\langle M_{n+1}, M_{n+1}\rangle(t)\right) = 0,$$

and thus $R_{n+1} = \int_0^{T^*} \psi_{n+1}(t) dM_{n+1}(t) = 0$ almost surely. Since $Z_1$ is by assumption independent of $Z^{1-}$ and thus independent of $Z_j$ for all $j = 2, \dots, n+1$, it follows that $\sigma_{1,\cdot}(t)\sigma_{j,\cdot}(t)^\top dt = d\langle M_1, Z_j\rangle(t) = d\langle Z_1, Z_j\rangle(t) = 0$ for all $j \notin \{1, n+1\}$. Thus, the matrix $A_t$ from above now equals

$$A_t := \begin{pmatrix} \sigma_{1,\cdot}(t)\sigma_{1,\cdot}(t)^\top & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \tilde{A}_t & \\ 0 & & & \end{pmatrix}, \ \tilde{A}_t := \begin{pmatrix} \sigma_{2,\cdot}(t)\sigma_{2,\cdot}(t)^\top & \dots & \sigma_{2,\cdot}(t)\sigma_{n,\cdot}(t)^\top \\ \vdots & & \vdots \\ \sigma_{n,\cdot}(t)\sigma_{2,\cdot}(t)^\top & \dots & \sigma_{n,\cdot}(t)\sigma_{n,\cdot}(t)^\top \end{pmatrix},$$

and since $0 \neq \det A_t = \sigma_{1,\cdot}(t)\sigma_{1,\cdot}(t)^\top \det \tilde{A}_t$, it must hold $\det \tilde{A}_t \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Furthermore, since $\langle M_1, Z_j\rangle(t) = 0$ (see above) and $\langle M_{n+1}, Z_j\rangle(t) = 0$ for all $j \notin \{1, n+1\}$, we obtain

$$\begin{aligned} 0 &= d\langle L, Z_j\rangle(t) = \sum_{k=1}^{n+1} \psi_k(t) d\langle M_k, Z_j\rangle(t) = \sum_{k=2}^{n} \psi_k(t) d\langle M_k, Z_j\rangle(t) \\ &= \sum_{k=2}^{n} \psi_k(t)\sigma_{k,\cdot}(t)\sigma_{j,\cdot}^\top(t) dt, \ \ j \notin \{1, n+1\}, \ 0 \leq t \leq T^*. \end{aligned}$$

Analogously to the linear system from above, we get $\tilde{A}_t^\top \tilde{\psi}_t = 0$, where $\tilde{\psi}_t = (\psi_2(t), \dots, \psi_n(t))^\top$. Since $\det \tilde{A}_t \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, it follows that $\tilde{\psi}_t = 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, and thus $R_j = \int_0^{T^*} \psi_j(t) dM_j(t) = 0$ almost surely for all $j \notin \{1, n+1\}$.

P4: Consider any permutation $\pi : \{1,\dots,n+1\} \to \{1,\dots,n+1\}$. Let $(L, Z_{\pi(1)},\dots,Z_{\pi(n+1)}) \overset{\text{MRT}}{\leftrightarrow} (\tilde{R}_1,\dots,\tilde{R}_{n+1})$ with $\tilde{R}_i = \int_0^{T^*} \tilde{\psi}_i(t)dM_{\pi(i)}(t),\ i = 1,\dots,n+1$, where $\tilde{\psi}_i$ are $\mathbb{F}$-predictable processes. Since

$$\sum_{i=1}^{n+1}\int_0^{T^*} \tilde{\psi}_i(t)dM_{\pi(i)}(t) = \sum_{i=1}^{n+1}\tilde{R}_i \overset{\text{P6}^*}{=} L \overset{\text{P6}^*}{=} \sum_{i=1}^{n+1} R_i = \sum_{i=1}^{n+1}\int_0^{T^*} \psi_i(t)dM_i(t)$$
$$= \sum_{i=1}^{n+1}\int_0^{T^*} \psi_{\pi(i)}(t)dM_{\pi(i)}(t),$$

it follows by the uniqueness of the MRT decomposition that $\tilde{\psi}_i = \psi_{\pi(i)}$ $\lambda \otimes \mathbb{P}$-almost surely, and thus $\tilde{R}_i = R_{\pi(i)}$ almost surely for all $i = 1,\dots,n+1$.

P5: Assume that $\tilde{Z}_i(t) := f_i(Z_i(t)),\ i = 1,\dots,n+1$, where the functions $f_i : \mathbb{R} \to \mathbb{R}$ are Borel measurable and invertible, and consider $(L, \tilde{Z}_1,\dots,\tilde{Z}_{n+1}) \overset{\text{MRT}}{\leftrightarrow} (\tilde{R}_1,\dots,\tilde{R}_{n+1})$.

(a) For each $i \neq n+1$, the function $f_i$ is by assumption twice continuously differentiable and it follows by Itô's lemma that

$$d\tilde{Z}_i(t) = df_i(X_i(t))$$
$$= f_i'(X_i(t))\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t) + \left( f_i'(X_i(t))\theta(t) + \frac{1}{2}f_i''(X_i(t))\sum_{j=1}^{d}\sigma_{ij}^2(t)\right) dt.$$

Thus, $(\tilde{Z}_1,\dots,\tilde{Z}_n)$ is again an Itô process as in Assumption 3.1, and by Lemma 3.2 the corresponding compensated risk processes equal

$$\tilde{M}_i(t) = f_i'(X_i(t))dM_i(t),\ i = 1,\dots,n.$$

As a result, for $i = 1,\dots,n$ the MRT risk factors are equal to

$$\tilde{R}_i = \int_0^{T^*} \tilde{\psi}_i(t)d\tilde{M}_i(t) = \int_0^{T^*} \tilde{\psi}_i(t)f_i'(X_i(t))dM_i(t). \tag{4.1}$$

(b) If $i = n+1$, we require that $\tilde{Z}_{n+1}$ is a again a counting process which fits to our setting. Otherwise, the MRT decomposition is not defined. First observe that

$$\tilde{Z}_{n+1}(t) = f_{n+1}(N(t))$$
$$= f_{n+1}(N(0)) + \sum_{0<s\leq t}\left(f_{n+1}(N(s)) - f_{n+1}(N(s-))\right)$$
$$= f_{n+1}(N(0))$$
$$+ \sum_{0<s\leq t}\left[\sum_{k=0}^{m}\mathbb{1}_{\{N(s-)=k\}}\left(f_{n+1}(k+1) - f_{n+1}(k)\right)\right](N(s) - N(s-))$$
$$= f_{n+1}(N(0)) + \int_0^t a(s)dN(s)$$
$$= f_{n+1}(N(0)) + \int_0^t a(s)dM_{n+1}(s) + \int_0^t a(s)(m - N(s-))\mu(s)ds,$$

where $a(s) := \sum_{k=0}^{m} \mathbb{1}_{\{N(s-)=k\}} (f_{n+1}(k+1) - f_{n+1}(k))$ is predictable so that $\tilde{A}_{n+1}(t) := \int_0^t a(s)(m - N(s-))\mu(s)ds$ is a predictable finite variation process and $\tilde{M}_{n+1}(t) := \int_0^t a(s)dM_{n+1}(s)$ is a local martingale. For the third equality we have used that in our framework $\mathbb{P}(\tau_x^i = \tau_x^j) = 0$ for $i \neq j$ (Bielecki and Rutkowski, 2004, p. 269). Then it follows that

$$\tilde{R}_{n+1} = \int_0^{T^*} \tilde{\psi}_{n+1}(t)d\tilde{M}_{n+1}(t) = \int_0^{T^*} \tilde{\psi}_{n+1}(t)a(t)dM_{n+1}(t). \tag{4.2}$$

The uniqueness of the MRT decomposition together with (4.1) and (4.2) imply that $R_i = \tilde{R}_i$ almost surely, $i = 1, \dots, n+1$.

□

Of course, the notion of uniqueness in Proposition 3.3 relies on the description of $X$ in Assumption 3.1 in terms of independent Brownian motions, which generate the filtration $\mathbb{G}$. However, such a description is not necessarily unique. In particular, if $X$ is specified in terms of correlated Brownian motions, which is frequently assumed in literature, a unique description in terms of independent Brownian motions is not guaranteed. In more detail, assume that $\mathbb{G}$ is the augmented filtration generated by a $d$-dimensional Brownian motion $B = (B_1, \dots, B_d)$ with (possibly) correlated one-dimensional Brownian motions $B_i$, $i = 1, \dots, d$, and assume that the sources of risk $X_1, \dots, X_n$ are specified in terms of this Brownian motion $B$, i.e.

$$dX(t) = \theta(t)dt + \sigma(t)dB(t).$$

It is well-known that, if

$$dB_i(t)dB_j(t) = \rho_{ij}(t)dt, \quad i,j = 1, \dots, d,$$

for some deterministic functions $\rho_{ij}(t)$ taking values in $(-1, 1)$ for $i \neq j$ and $\rho_{ij}(t) = 1$ for $i = j$, and if the symmetric matrix $\rho(t) = (\rho_{ij}(t))_{i,j=1,\dots,d}$ is positive definite for all $t$, then there exist $\mathbb{G}$-adapted independent Brownian motions $\tilde{W}_1(t), \dots, \tilde{W}_d(t)$ and a matrix $A(t) = (a_{ij}(t))_{i,j=1,\dots,d}$ of deterministic functions $a_{ij}(t)$, which satisfies $\rho(t) = A(t)A(t)^\top$ and is thus invertible for all $t$, such that

$$B_i(t) = \sum_{j=1}^{d} \int_0^t a_{ij}(s)d\tilde{W}_j(s), \quad j = 1, \dots, d$$

(Exercise 4.16 in Shreve, 2004, p. 200). Note that the augmented natural filtrations generated by $B$ and $\tilde{W}$, respectively, coincide. The SDEs of the processes $(X_i(t))_{0 \le t \le T^*}$ can then be rewritten as

$$dX_i(t) = \theta_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dB_j(t) = \theta_i(t)dt + \sum_{k=1}^{d} \underbrace{\sum_{j=1}^{d} \sigma_{ij}(t)a_{jk}(t)}_{=:\tilde{\sigma}_{ik}(t)} d\tilde{W}_k(t), \quad i = 1, ...n, \tag{4.3}$$

and the situation of Assumption 3.1 is reestablished. However, the matrix $A(t)$ and thus the representation of (possibly) correlated Brownian motions by means of independent Brownian motions is not unique, which transfers to the martingale representation consisting of integrals with respect to independent Brownian motions. In contrast, we will show in the next corollary that the MRT decomposition formulated in terms of compensated risk processes is unique even in case of correlated Brownian motions. In particular, the MRT decomposition is independent of how we actually model correlations between Brownian motions which is another argument in favor of our decomposition approach. Actually, in contrast to variance and Hoeffding decomposition, the MRT decomposition would yield in Example 2.2 the expected result $R_1 = B_1(T)$. All these aspects are illustrated in Example 4.3.

**Corollary 4.2.** *The statement of Proposition 3.3 is still true if $W = (W_1, \ldots, W_d)$ is a $d$-dimensional Brownian motion with correlated one-dimensional Brownian motions $W_i,\ i = 1, \ldots, d$, given the correlation matrix $\rho(t) = (\rho_{ij}(t))_{i,j=1,\ldots,d}$ is symmetric and positive definite for all $t \in [0, T^*]$ almost surely, and the functions $\rho_{ij}(t)$ are deterministic taking values in $(-1, 1)$ for $i \neq j$ and $\rho_{ij}(t) = 1$ for $i = j$.*

*Proof.* The existence of the decomposition directly follows by (4.3) and the foregoing discussion, together with Proposition 3.3. The conditions of Proposition 3.3 are satisfied since $\det \widetilde{\sigma}_{ik}(t) = \det \sigma(t) \det A(t) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. Although the representation in (4.3) is not unique, the uniqueness of the MRT decomposition still follows by the uniqueness from Proposition 3.3 since the compensated risk processes $M_i^W,\ i = 1, \ldots, n$, are the same for each representation. □

**Example 4.3.** *As in Example 2.2 let $L_0 = B_1(T) + 2B_2(T)$, where $B_1$ and $B_2$ are two one-dimensional Brownian motions with $dB_1(t)dB_2(t) = \rho dt,\ \rho \in (-1, 1)\backslash\{0, -0.8\}$. Then it can be shown that*

$$\begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix} = \begin{pmatrix} W_1(t) \\ \rho W_1(t) + \sqrt{1-\rho^2} W_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix},$$

*for some independent Brownian motions $W_1$ and $W_2$, as well as (switching the roles of $B_1$ and $B_2$)*

$$\begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix} = \begin{pmatrix} \rho W_3(t) + \sqrt{1-\rho^2} W_4(t) \\ W_3(t) \end{pmatrix} = \begin{pmatrix} \rho & \sqrt{1-\rho^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_3(t) \\ W_4(t) \end{pmatrix},$$

*for some independent Brownian motions $W_3$ and $W_4$. Obviously, the two versions imply the two martingale representations with respect to independent Brownian motions*

$$\begin{aligned} L_0 - \mathrm{E}^{\mathbb{P}}(L_0) &= \int_0^T (1 + 2\rho) dW_1(t) + \int_0^T 2\sqrt{1-\rho^2}\, dW_2(t) \\ &= (1 + 2\rho) W_1(T) + 2\sqrt{1-\rho^2} W_2(T) \end{aligned}$$

*and*

$$\begin{aligned} L_0 - \mathrm{E}^{\mathbb{P}}(L_0) &= \int_0^T (\rho + 2) dW_3(t) + \int_0^T \sqrt{1-\rho^2}\, dW_4(t) \\ &= (\rho + 2) W_3(T) + \sqrt{1-\rho^2} W_4(T), \end{aligned}$$

*respectively. Since e.g.* $\mathrm{Var}((1 + 2\rho)W_1(T)) \neq \mathrm{Var}((\rho + 2)W_3(T))$ *and* $\mathrm{Var}((1 + 2\rho)W_1(T)) \neq \mathrm{Var}(\sqrt{1-\rho^2} W_4(T))$ *for the considered $\rho$, the two martingale representations with respect to independent Brownian motions are not the same (in the almost sure sense). In contrast, the MRT decomposition is unique and in both cases equal to*

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \int_0^T 1\, dB_1(t) + \int_0^T 2 dB_2(t) = B_1(T) + 2B_2(T),$$

*which is also what we actually would expect.*

## 4.2 Calculation of the MRT decomposition

### General case

As seen in the proof of Proposition 3.3, when $n = d$ and $\det \sigma(t) \neq 0,\ t \in [0, T^*]$, it is sufficient to determine the integrands $\widetilde{\psi}_1^W, \ldots, \widetilde{\psi}_n^W, \psi^N$ resulting from the martingale representation theorem. The integrands $\psi_1^W, \dots, \psi_n^W, \psi^N$ of the MRT decomposition then directly follow. However, as Björk (2005, p. 157) states in general we know very little about the exact form of the processes $\widetilde{\psi}_1^W, \dots, \widetilde{\psi}_n^W, \psi^N$. The most precise description is provided by the theory of Malliavin calculus, in particular by the so-called Clark-Ocone formula (for an introduction to Malliavin calculus, see Nualart, 2006). However, the Clark-Ocone formula is only applicable to independent driving processes, and as soon as mortality intensities are modeled stochastically, such an independence is usually no longer given between the number of deaths in the portfolio $N$ and the standard Brownian motion $W$ driving among others the mortality intensity. Thus, the following three lemmas reduce the problem to finding the martingale representation of a $\mathbb{G}$-martingale instead of an $\mathbb{F}$-martingale, i.e. the problem is reduced to a Brownian motion setting.

The first lemma covers discrete survival cash flows.

**Lemma 4.4.** *Let $Z$ be a random variable of the form $Z = (m - N(T))\, F,\ 0 \leq T \leq T^*$, where $F$ is a $\mathcal{G}_{T^*}$-measurable and integrable random variable. Then there exist predictable processes $\varphi_1, \dots, \varphi_d$ such that*

$$\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(T)}F\middle|\, \mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(T)}F\right) + \sum_{i=1}^{d}\int_0^t \varphi_i(u)dW_i(u),\ \ t \leq T^*, \tag{4.4}$$

*and the martingale representation of $Z$ is given by*

$$\begin{aligned} Z =\ & \mathrm{E}^{\mathbb{P}}(Z) + \sum_{i=1}^{d}\int_0^{T^*} \left[(m - N(t-))\, e^{\Gamma(t)}\mathbb{1}_{[0,T]}(t) + (m - N(T))\, e^{\Gamma(T)}\mathbb{1}_{(T,T^*]}(t)\right] \varphi_i(t)dW_i(t) \\ & - \int_{0+}^{T} \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(T)}F\middle|\mathcal{G}_t\right) dM^N(t). \end{aligned} \tag{4.5}$$

**Remark 4.5.** *Assume that $X$ is an $n$-dimensional Itô process as specified in Assumption 3.1 with compensated risk processes $M_i^W$ as given in Lemma 3.2. Then the previous lemma still holds when in both equations* (4.4) *and* (4.5) *the Brownian motions $W_j,\ j = 1, \dots, d$, are replaced by the martingales $M_i^W,\ i = 1, \dots, n$ (when at the same time the summation and the size of the vector $\varphi$ are adapted). This will be exploited in the proof of Proposition 4.13 ii). It follows from Lemma 4.4 and using, before and after applying the lemma, the equality*

$$\sum_{i=1}^{n}\int_0^t \tilde{\varphi}_i(u)dM_i^W(u) = \int_0^t \tilde{\varphi}(u)dM^W(u) = \int_0^t \tilde{\varphi}(u)\sigma(u)dW(u) = \sum_{j=1}^{d}\int_0^t (\tilde{\varphi}(u)\sigma(u))_j dW_j(u),$$

*where $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_n),\ M^W = (M_1^W, \dots, M_n^W)$ and $(\dots)_i$ denotes the $i$-th component of a vector.*

For a single policyholder, the proof of Lemma 4.4 given in the Appendix A mainly follows the ideas of the proof of Proposition 5.2.2 in Bielecki and Rutkowski (2004, pp. 159). We further modify their result so that it fits to our later application and extend it to an entire (homogeneous) portfolio. In addition, there seems to be a typo in the proof with respect to the integrands of the $dW_i$-terms which we correct here. For $F$ $\mathcal{G}_T$-measurable instead of (more generally) $\mathcal{G}_{T^*}$-measurable, similar results (with or without proof, usually in a specific process setting) have been derived in the context of risk-minimizing hedging strategies, see e.g. Barbarin (2008, Prop. 4.10, Prop. 5.11), Biagini et al. (2012, Prop. 3.5), Biagini et al. (2013, Prop. 2, Prop. 9), and Biagini and Schreiber (2013, Lemma 4.2). In particular, most of them also consider entire portfolios.

The next lemma covers the case of continuous survival cash flows.

**Lemma 4.6.** *Let $Z$ be a random variable of the form $Z = \int_0^T (m - N(v))F(v)dv,\ 0 \le T \le T^*$, where $F = (F(t))_{0\le t\le T}$ is a $\mathbb{G}$-predictable process with $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}|F(t)|\right) < \infty$. Then there exist predictable processes $\varphi_1, \ldots, \varphi_d$ such that*

$$\mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)}F(v)dv \middle| \mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)}F(v)dv\right) + \sum_{i=1}^d \int_0^t \varphi_i(u)dW_i(u),\ \ t \le T, \tag{4.6}$$

*and the martingale representation of $Z$ is given by*

$$\begin{aligned} Z = \mathrm{E}^{\mathbb{P}}(Z) &+ \sum_{i=1}^d \int_0^T (m - N(t-))e^{\Gamma(t)}\varphi_i(t)dW_i(t) \\ &- \int_{0+}^T \mathrm{E}^{\mathbb{P}}\left(\int_t^T e^{\Gamma(t)-\Gamma(v)}F(v)dv\big|\mathcal{G}_t\right) dM^N(t). \end{aligned} \tag{4.7}$$

For the sake of completeness, a proof is given in Appendix A. Except of some details, it mainly relies on the proof of Proposition 4.12 in Barbarin (2008).

**Remark 4.7.** *i) Similarly as for Lemma 4.4, under Assumption 3.1 it can be shown that the previous lemma still holds when in both equations the Brownian motions $W_j$ are replaced by the martingales $M_i^W$.*

*ii) Assume that for every $v \in [0, T]$*

$$\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(v)}F(v)\big| \mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(v)}F(v)\right) + \sum_{i=1}^d \int_0^t \varphi_i^v(u)dW_i(u),\ \ t \le v,$$

*for some predictable processes $\varphi_1^v, \ldots, \varphi_d^v$. If $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left([F(t)]^2\right) < \infty$, it can be shown via the stochastic Fubini theorem (Protter, 2005, Theorem 65) that $\varphi_i$ in* (4.6) *equals*

$$\varphi_i(u) = \int_u^T \varphi_i^v(u)dv,\ \ u \le T.$$

*This may simplify the derivation of* (4.6) *in some cases. For bounded $F$, this result has already be shown in Biagini et al. (2013, Proposition 5).*

*iii) Since we assume that $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}|F(t)|\right) < \infty$, it follows by the theorem of Fubini-Tonelli for conditional expectations that conditional expectation and integral in the $dM^N$-integrand of* (4.7) *can be interchanged.*

The next lemma covers continuous cash flows contingent on death.

**Lemma 4.8.** *Let $Z$ be a random variable of the form $Z = \int_0^T F(v)dN(v),\ 0 \le T \le T^*$, where $F = (F(t))_{0\le t\le T}$ is a continuous and $\mathbb{G}$-predictable process with $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}|F(t)|\right) < \infty$. Then there exist predictable processes $\varphi_1, \ldots, \varphi_d$ such that for $t \le T$*

$$\mathrm{E}^{\mathbb{P}}\left(\int_0^T F(v)e^{-\Gamma(v)}d\Gamma(v) \middle| \mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(\int_0^T F(v)e^{-\Gamma(v)}d\Gamma(v)\right) + \sum_{i=1}^d \int_0^t \varphi_i(u)dW_i(u), \tag{4.8}$$

*and the martingale representation of $Z$ is given by*

$$\begin{aligned} Z = \mathrm{E}^{\mathbb{P}}(Z) &+ \sum_{i=1}^d \int_0^T (m - N(t-))\, e^{\Gamma(t)}\varphi_i(t)dW_i(t) \\ &- \int_{0+}^T \left[\mathrm{E}^{\mathbb{P}}\left(\int_t^T F(v)e^{\Gamma(t)-\Gamma(v)}d\Gamma(v) \middle| \mathcal{G}_t\right) - F(t)\right] dM^N(t). \end{aligned} \tag{4.9}$$

A proof that relies on a generalization of Proposition 4 in Biagini et al. (2013, p. 130, 138) (which is actually based on Proposition 4.11 in Barbarin (2008)) is provided in Appendix A. Similar results were independently derived in Section 3.3 of Biagini et al. (2012) and in Section 4 of Biagini and Schreiber (2013).

**Remark 4.9.** *i) Part i) and iii) of Remark 4.7 also apply to Lemma 4.8, where for part iii)* $d\Gamma(t)$ *in* (4.9) *is replaced by* $\mu(t)dt$.

*ii) Similarly as in Remark 4.7 ii), assume that for every* $v \in [0,T]$

$$\mathrm{E}^{\mathbb{P}}\left(F(v)e^{-\Gamma(v)}\mu(v)\middle|\,\mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(F(v)e^{-\Gamma(v)}\mu(v)\right) + \sum_{i=1}^{d}\int_0^t \varphi_i^v(u)dW_i(u), \;\; t \le v,$$

*for some predictable processes* $\varphi_1^v, \ldots, \varphi_d^v$. *If* $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left([F(t)]^4\right) < \infty$ *and* $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^4(t)\right) < \infty$, *it can again be shown via the stochastic Fubini theorem (Protter, 2005, Theorem 65) and using* $\Gamma(t) = \int_0^t \mu(s)ds$ *that* $\varphi_i$ *in* (4.8) *also equals* $\varphi_i(u) = \int_u^T \varphi_i^v(u)dv, \;\; u \le T$.

Combining the previous three lemmas (and the related remarks) with the Clark-Ocone formula from Malliavin calculus, we obtain the MRT decomposition of each summand of $L_0$ defined in Section 3.1. Clearly, the MRT decomposition of $L_0$ itself then follows by summing up the individual decompositions. Note that the previous three lemmas did not require that $X$ is an Itô process. However, since we aim at a decomposition with respect to the compensated risk processes, this assumption needs to be added at this stage.

In what follows, let $\mathbb{D}_{1,2}$ denote the set of random variables that are Malliavin differentiable with respect to each one-dimensional Brownian motion $W_i$ of $W = (W_1, \ldots, W_d)$, and let $D_{t,i}(\cdot)$ denote the respective time-$t$ Malliavin derivative with respect to $W_i, \; i = 1, \ldots, d$. For a definition of Malliavin differentiability and Malliavin derivative, we refer to Definition 3.1 in Di Nunno et al. (2009, p. 27). Note that all random variables in $\mathbb{D}_{1,2}$ are by definition in $L^2(\mathbb{P})$ and $\mathcal{G}_{T^*}$-measurable. A general introduction to Malliavin calculus can be found in Di Nunno et al. (2009) or Nualart (2006). A summary with helpful results is given, among others, by Fournié et al. (1999), Benhamou (2002) and Peng and Hu (2013).

**Proposition 4.10.** *Let* $X$ *be an* $n$*-dimensional Itô process as specified in Assumption 3.1. Assume that* $n = d$ *and that the inverse* $\sigma^{-1}(t) = \left(\sigma_{ij}^{-1}(t)\right)_{i,j=1,\ldots,n}$ *exists for all* $t \in [0,T^*]$ $\mathbb{P}$*-almost surely. Let* $0 \le t_k \le T \le T^*$.

*i) Let* $L_0 = C_0$. *If* $C_0 \in \mathbb{D}_{1,2}$, *then the unique integrands of the MRT decomposition* (3.3) *of* $L_0$ *are given by*

$$\psi_i^W(t) = \sum_{j=1}^{d} \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(C_0\right)\middle|\,\mathcal{G}_t\right)\sigma_{ji}^{-1}(t), \; i = 1, \ldots, n,$$

$$\psi^N(t) = 0.$$

*ii) Let* $L_0 = (m - N(t_k))C_{a,k}$. *If* $C_{a,k} \in \mathbb{D}_{1,2}$, *and* $e^{\Gamma(t)} \in \mathbb{D}_{1,2}$ *for all* $t \in [0,t_k]$, *then the unique integrands of the MRT decomposition* (3.3) *of* $L_0$ *are given by*

$$\psi_i^W(t) = \left[(m - N(t-))e^{\Gamma(t)}\mathbb{1}_{[0,t_k]}(t) + (m - N(t_k))e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T^*]}(t)\right]$$
$$\times \sum_{j=1}^{d} \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\middle|\,\mathcal{G}_t\right)\sigma_{ji}^{-1}(t), \;\; i = 1, \ldots, n,$$

$$\psi^N(t) = -\mathbb{1}_{[0,t_k]}(t)\,\mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\,\mathcal{G}_t\right).$$

*iii) Let $L_0 = \int_0^T (m - N(t))C_a(t)dt$. If $C_a = (C_a(t))_{0\le t\le T}$ is a $\mathbb{G}$-predictable process with $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}[C_a(t)]^2\right) < \infty$, and $C_a(t)$, $e^{\Gamma(t)} \in \mathbb{D}_{1,2}$ for all $t \in [0,T]$, then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by for $i = 1,\dots,n$,*

$$\psi_i^W(t) = \mathbb{1}_{[0,T]}(t)\,(m - N(t-))e^{\Gamma(t)} \sum_{j=1}^d \int_t^T \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(e^{-\Gamma(v)}C_a(v)\right)\middle|\,\mathcal{G}_t\right) dv\,\sigma_{ji}^{-1}(t),$$

$$\psi^N(t) = -\mathbb{1}_{[0,T]}(t)\int_t^T \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(v)}C_a(v)\middle|\mathcal{G}_t\right) dv.$$

*iv) Let $L_0 = \int_0^T C_{ad}(t)dN(t)$. If $C_{ad} = (C_{ad}(t))_{0\le t\le T}$ is a continuous and $\mathbb{G}$-predictable process with $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}|C_{ad}(t)|\right) < \infty$, $e^{\Gamma(t)} \in \mathbb{D}_{1,2}$ for all $t \in [0,T]$, and $\int_0^T C_{ad}(t)e^{-\Gamma(t)}d\Gamma(t) \in \mathbb{D}_{1,2}$,then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by for $i = 1,\dots,n$,*

$$\psi_i^W(t) = \mathbb{1}_{[0,T]}(t)\,(m - N(t-))e^{\Gamma(t)} \sum_{j=1}^d \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(\int_0^T C_{ad}(v)e^{-\Gamma(v)}d\Gamma(v)\right)\middle|\,\mathcal{G}_t\right)\sigma_{ij}^{-1}(t),$$

$$\psi^N(t) = -\mathbb{1}_{[0,T]}(t)\left[\mathrm{E}^{\mathbb{P}}\left(\int_t^T C_{ad}(v)e^{\Gamma(t)-\Gamma(v)}d\Gamma(v)\middle|\,\mathcal{G}_t\right) - C_{ad}(t)\right].$$

*Proof.* The integrands directly follow from Lemma 4.4, Lemma 4.6 including the related remark, and Lemma 4.8 together with the Clark-Ocone formula, which can be found in Di Nunno et al. (2009, p. 196), and the proof of Proposition 3.3. The uniqueness also follows from Proposition 3.3. □

We omitted to determine the MRT decomposition of discrete cash flows contingent on death since

$$(N(t_k) - N(t_{k-1}))\,C_{ad,k} = (m - N(t_{k-1}))C_{ad,k} - (m - N(t_k))C_{ad,k},$$

i.e. they can be represented as a sum of two discrete survival cash flows. The MRT decompositions of these two summands can then be determined via Proposition 7 ii). Note that therein we do not require that $C_{a,k}$ is $\mathcal{G}_{t_k}$-measurable.

An application of the above proposition is given in the following example, where the MRT decomposition of a pure endowment portfolio is determined.

**Example 4.11.** *Consider a portfolio of $m$ pure endowment policies with survival benefit $1$ at time $T$ and single premium $P_0$ at time $0$. In order to keep the example simple, we do not discount the survival benefit, so that the insurer's time-$0$ loss equals $L_0 = -mP_0 + (m - N(T))$. The random lifetimes are modeled as described in Section 3.1. The mortality intensity is assumed to be an affine diffusion process (Biffis, 2005), i.e.*

$$d\mu(t) = \theta(t,\mu(t))dt + \sigma(t,\mu(t))dW(t),\ \mu(0) = \mu_0,$$

*and (Björk, 2005, Proposition 22.2)*

$$\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T \mu(s)ds}\middle|\,\mathcal{G}_t\right) = e^{\alpha(t)+\beta(t)\mu(t)},\ T \in (t, T^*], \tag{4.10}$$

*where $\alpha$ and $\beta$ satisfy certain Riccati ordinary differential equations. Clearly, since $-mP_0$ is deterministic, the integrands of its MRT decomposition are zero. Assume that $\mu(t)$ is non-negative, $\theta(t,x)$ and*

*$\sigma(t,x)$ are continuous in $x$, $\det\sigma(t,\mu(t)) \neq 0$ for all $t \in [0,T]$ $\mathbb{P}$-almost surely, and $e^{\Gamma(t)} \in \mathbb{D}_{1,2}$ for all $t \in [0,T]$. Then it follows by applying part ii) of Proposition 4.10 to $(m - N(T))$ that*

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \int_0^T (m - N(t-))e^{\Gamma(t)} \frac{\mathrm{E}^{\mathbb{P}}\left(D_t\left(e^{-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right)}{\sigma(t,\mu(t))} dM^W(t) - \int_{0+}^T \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(T)}\middle|\,\mathcal{G}_t\right) dM^N(t).$$

*Since $D_{t,j}(\mu(s)) = 0$ for all $t > s$, and thus by the chain rule $D_{t,j}\left(e^{\Gamma(t)}\right) = 0$, the product rule (Di Nunno et al., 2009, p. 30) implies*

$$D_{t,j}\left(e^{\Gamma(t)-\Gamma(T)}\right) = e^{\Gamma(t)}D_{t,j}\left(e^{-\Gamma(T)}\right) + e^{-\Gamma(T)}D_{t,j}\left(e^{\Gamma(t)}\right) = e^{\Gamma(t)}D_{t,j}\left(e^{-\Gamma(T)}\right),$$

*i.e. $e^{\Gamma(t)}\mathrm{E}^{\mathbb{P}}\left(D_t\left(e^{-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(D_t\left(e^{\Gamma(t)-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right)$. Furthermore, exchanging conditional expectation and Malliavin derivative operator (Di Nunno et al., 2009, Proposition 3.12, p. 33) together with* (4.10) *we have*

$$\mathrm{E}^{\mathbb{P}}\left(D_t\left(e^{\Gamma(t)-\Gamma(T)}\right)\middle|\,\mathcal{G}_t\right) = D_t\left(\mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(T)}\middle|\,\mathcal{G}_t\right)\right) = D_t\left(e^{\alpha(t)+\beta(t)\mu(t)}\right).$$

*If we additionally assume that the conditions from Theorem 2.2.1 in Nualart (2006, p. 119) on the diffusion coefficients $\theta$ and $\sigma$ are satisfied, it follows that the Malliavin derivative of $\mu(t)$ is given by $D_t(\mu(t)) = \sigma(t,\mu(t))$, and the chain rule from Malliavin calculus (León, 2003, Lemma 2.1, p. 174) yields*

$$D_t\left(e^{\alpha(t)+\beta(t)\mu(t)}\right) = e^{\alpha(t)+\beta(t)\mu(t)}\beta(t)D_t(\mu(t)) = e^{\alpha(t)+\beta(t)\mu(t)}\beta(t)\sigma(t,\mu(t)).$$

*Summing up, we obtain*

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \int_0^T (m - N(t-))\, e^{\alpha(t)+\beta(t)\mu(t)}\beta(t)dM^W(t) - \int_{0+}^T e^{\alpha(t)+\beta(t)\mu(t)}dM^N(t),$$

*where the first summand represents the systematic mortality risk and the second summand the unsystematic mortality risk.*

### Markov case

Given the cash flows have a certain structure, we can describe the integrands of the MRT decomposition more explicitly. More precisely, in what follows we assume that $X$ is a diffusion process (and thus Markov) and that the insurance payments are specific functions of the state variables. In this case, we can directly evaluate the decompositions via Itô's formula rather than relying on Malliavin derivatives as in Proposition 4.10.[5]

In what follows, we write $f \in C^{1,2}([0,T] \times \mathbb{R}^n)$ for a function $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ if the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $1 \leq i,j \leq n$, exist, are continuous on $(0,T) \times \mathbb{R}^n$, and have continuous extensions to $[0,T] \times \mathbb{R}^n$. With respect to notation, we do not explicitly distinguish between the derivatives and their continuous extensions.

[5]Note that for globally Lipschitz-continuous coefficients $\theta$ and $\sigma$ with at most linear growth, diffusion processes are Malliavin differentiable (Nualart, 2006, Theorem 2.2.1, p. 119). However, as the discussion on the Malliavin differentiability of square-root processes shows (Alòs and Ewald, 2008), the general Malliavin differentiability of diffusion processes is not guaranteed.

**Assumption 4.12.** *The state process $X = ((X_1(t), \dots, X_n(t))^\top)_{0\le t\le T^*}$ is an $n$-dimensional diffusion process satisfying*

$$dX(t) = \theta(t, X(t))dt + \sigma(t, X(t))dW(t), \tag{4.11}$$

*with deterministic initial value $X(0) = x_0 \in \mathbb{R}^n$, where the drift vector $\theta : [0,T^*] \times \mathbb{R}^n \to \mathbb{R}^n$ and the volatility matrix $\sigma : [0,T^*] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are continuous functions such that a unique strong solution exists to* (4.11).

**Proposition 4.13.** *Let $X$ be an $n$-dimensional diffusion process as specified in Assumption 4.12 and assume that $n = d$ and that $\det \sigma(t, X(t)) \neq 0$ for all $t \in [0,T^*]$ $\mathbb{P}$-almost surely. Let $0 \le t_k \le T^*$, $0 \le T \le T^*$.*

*i) Let $L_0 = C_0$. Assume that $C_0$ is integrable and of the form*

$$C_0 = e^{-\int_0^T g(s,X(s))ds} h(X(T))$$

*for some measurable functions $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to [0,\infty)$. Define $f(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right)$. If $f \in C^{1,2}([0,T] \times \mathbb{R}^n)$, then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by*

$$\psi_i^W(t) = \mathbb{1}_{[0,T]}(t)\, e^{-\int_0^t g(s,X(s))ds} \frac{\partial f}{\partial x_i}(t, X(t)),\ i = 1,\dots,n,$$
$$\psi^N(t) = 0.$$

*ii) Let $(m - N(t_k))C_{a,k}$. Assume that $C_{a,k}$ is integrable and of the form*

$$C_{a,k} = e^{-\int_0^T g(s,X(s))ds} h(X(T))$$

*for some measurable functions $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to [0,\infty)$.*

*(a) Assume that $T > t_k$, and define $f^A : [0,t_k] \times \mathbb{R}^n \to \mathbb{R}$ and $f^B : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ by*

$$f^A(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^{t_k} \mu(s,X(s))ds} e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right),$$
$$f^B(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right).$$

*If $f^A \in C^{1,2}([0,t_k] \times \mathbb{R}^n)$ and $f^B \in C^{1,2}([0,T] \times \mathbb{R}^n)$, then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by*

$$\psi_i^W(t) = \mathbb{1}_{[0,t_k]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds} \frac{\partial f^A}{\partial x_i}(t, X(t))$$
$$+ \mathbb{1}_{(t_k,T]}(t)\,(m - N(t_k))e^{-\int_0^t g(s,X(s))ds} \frac{\partial f^B}{\partial x_i}(t, X(t)),\ \ i = 1,\dots,n,$$
$$\psi^N(t) = -\mathbb{1}_{[0,t_k]}(t)\, e^{-\int_0^t g(s,X(s))ds} f^A(t, X(t)).$$

*(b) Assume that $T \le t_k$, and define $f^A : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $f^B : [0,t_k] \times \mathbb{R}^n \to \mathbb{R}$ by*

$$f^A(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^{t_k} \mu(s,X(s))ds} e^{-\int_t^T g(s,X(s))ds} h(X(T)) \middle| X(t) = x\right),$$
$$f^B(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^{t_k} \mu(s,X(s))ds} \middle| X(t) = x\right).$$

*If $f^A \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and $f^B \in C^{1,2}([0,t_k] \times \mathbb{R}^n)$, then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by*

$$\begin{aligned}
\psi_i^W(t) &= \mathbb{1}_{[0,T]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds}\frac{\partial f^A}{\partial x_i}(t, X(t)) \\
&\quad + \mathbb{1}_{(T,t_k]}(t)\,(m - N(t-))C_{a,k}\frac{\partial f^B}{\partial x_i}(t, X(t)), \;\; i = 1, \dots, n, \\
\psi^N(t) &= -\mathbb{1}_{[0,T]}(t)\,e^{-\int_0^t g(s,X(s))ds} f^A(t, X(t)) - \mathbb{1}_{(T,t_k]}\,C_{a,k} f^B(t, X(t)).
\end{aligned}$$

*iii) Let $L_0 = \int_0^T (m - N(v))F(v)dv$. Assume that $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}[C_a(t)]^2\right) < \infty$, and that $C_a(t)$ is of the form*

$$C_a(t) = e^{-\int_0^t g(s,X(s))ds} h(X(t))$$

*for some measurable functions $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to [0,\infty)$. Define $f^v(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^v [g(s,X(s))+\mu(s,X(s))]ds} h(X(v)) \middle| X(t) = x\right)$. If $f^v \in C^{1,2}([0,v] \times \mathbb{R}^n)$ for all $v \in [0,T]$, then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by*

$$\begin{aligned}
\psi_i^W(t) &= \mathbb{1}_{[0,T]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds}\int_t^T \frac{\partial f^v}{\partial x_i}(t, X(t))dv, \;\; i = 1, \dots, n, \\
\psi^N(t) &= -\mathbb{1}_{[0,T]}(t)\,e^{-\int_0^t g(s,X(s))ds}\int_t^T f^v(t, X(t))dv.
\end{aligned}$$

*iv) Let $L_0 = \int_0^T C_{ad}(t)dN(t)$. Assume that $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]}|C_{ad}(t)|\right) < \infty$, and that $C_{ad}(t)$ is of the form*

$$C_{ad}(t) = e^{-\int_0^t g(s,X(s))ds} h(t, X(t))$$

*for some measurable and continuous functions $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $h : [0,T] \times \mathbb{R}^n \to [0,\infty)$. Define $f(t,x) := \mathrm{E}^{\mathbb{P}}\left(\int_t^T e^{-\int_t^s [g(v,X(v))+\mu(v,X(v))]dv} h(s, X(s))\mu(s)ds \middle| X(t) = x\right)$. If $f \in C^{1,2}([0,T] \times \mathbb{R}^n)$, then the unique integrands of the MRT decomposition* (3.3) *of $L_0$ are given by*

$$\begin{aligned}
\psi_i^W(t) &= \mathbb{1}_{[0,T]}(t)\,(m - N(t-))e^{-\int_0^t g(s,X(s))ds}\frac{\partial f}{\partial x_i}(t, X(t)), \;\; i = 1, \dots, n, \\
\psi^N(t) &= \mathbb{1}_{[0,T]}(t)\,\left[C_{ad}(t) - f(t, X(t))e^{-\int_0^t g(s,X(s))ds}\right].
\end{aligned}$$

The proof of Proposition 4.13 uses Lemma 4.4, Lemma 4.6, and 4.8 together with Itô's Lemma, and is provided in Appendix A. Note that even if $n \neq d$ this proof would imply the existence of the stated MRT decompositions. However, uniqueness is in general no longer guaranteed. For illustrating Proposition 4.13, we give the following example.

**Example 4.14.** *We again consider the setting from Example 4.11 but now determine the MRT decomposition of $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ by applying Proposition 4.13. Obviously, the mortality intensity is in this setting a one-dimensional diffusion process. Assuming a survival benefit and a discount factor of one, respectively, we have that $C_a(T) = e^{-\int_0^T g(s,X(s))ds} h(X(T))$ for $g \equiv 0$ and $h \equiv 1$. The affine property of the mortality model yields*

$$\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T [g(s,X(s))+\mu(s,X(s))]ds} h(X(T)) \middle| \mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T \mu(v)dv} \middle| \mathcal{G}_t\right) = e^{\alpha(t)+\beta(t)\mu(t)} =: f(t, \mu(t)).$$

*Since the function $f$ obviously satisfies the smoothness requirements, part b) of Proposition 4.13 ii) yields the MRT decomposition*

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \int_0^T (m - N(t-))\, e^{\alpha(t)+\beta(t)\mu(t)}\beta(t)dM^W(t) - \int_{0+}^T e^{\alpha(t)+\beta(t)\mu(t)}dM^N(t),$$

*where $-mP_0$ again does not contribute to the integrands since it is deterministic. This confirms the result from Example 4.11.*

To verify that the functions $f$ satisfy the required smoothness conditions imposed in Proposition 4.13, one can for instance rely on the (sufficient) conditions in Heath and Schweizer (2000). Of course, in case an analytic expression cannot be determined, the respective function $f$ can be computed numerically via Monte Carlo or by numerically solving the corresponding partial differential equations given by the Feynman-Kac theorem.

The following Proposition illustrates the relation between the integrands from Proposition 4.10 and Proposition 4.13, and generalizes the last part of Example 4.11.

**Proposition 4.15.** *Let $X$ be an $n$-dimensional diffusion process as specified in Assumption 4.12. Assume that $n = d$ and that the inverse $\sigma^{-1}(t, X(t)) = \left(\sigma_{ij}^{-1}(t, X(t))\right)_{i,j=1,\ldots,n}$ exists for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely. If $X_i(t) \in \mathbb{D}_{1,2}$ with $D_{t,j}X_k(t) = \sigma_{kj}(t, X(t))$ for all $i = 1, \ldots, n,\ t \in [0, T^*]$, and if $f : [0, T^*] \times \mathbb{R}^n \to \mathbb{R},\ (t, x) \mapsto f(t, x)$, is a continuously differentiable function with bounded partial derivatives, then for all $t \in [0, T^*]$ we have $f(t, X(t)) \in \mathbb{D}_{1,2}$ and*

$$\sum_{j=1}^d D_{t,j} f(t, X(t))\sigma_{ji}^{-1}(t, X(t)) = \frac{\partial}{\partial x_i} f(t, X(t)) \quad \forall i = 1, \ldots, n.$$

*Proof.* Using the chain rule (Proposition 1.2.3 in Nualart, 2006, p. 28), and $D_{t,j}t = 0$ (Theorem 2.2.1 in Nualart, 2006, p. 119), it follows that

$$\begin{aligned} D_{t,j} f(t, X(t)) &= \frac{\partial}{\partial t} f(t, X(t))D_{t,j}t + \sum_{k=1}^n \frac{\partial}{\partial x_k} f(t, X(t))D_{t,j}X_k(t) \\ &= \sum_{k=1}^n \frac{\partial}{\partial x_k} f(t, X(t))\sigma_{kj}(t, X(t)) \in \mathbb{D}_{1,2}. \end{aligned}$$

This together with $\sum_{j=1}^d \sigma_{kj}(t, X(t))\sigma_{ji}^{-1}(t, X(t)) = \mathbb{1}_{\{k=i\}}$ implies that

$$\begin{aligned} \sum_{j=1}^d D_{t,j} f(t, X(t))\sigma_{ji}^{-1}(t, X(t)) &= \sum_{k=1}^n \frac{\partial}{\partial x_k} f(t, X(t)) \sum_{j=1}^d \sigma_{kj}(t, X(t))\sigma_{ji}^{-1}(t, X(t)) \\ &= \frac{\partial}{\partial x_i} f(t, X(t)). \end{aligned}$$

□

## 4.3 Diversification properties

It is well known that unsystematic mortality risk arising from finite insurance portfolios vanishes as the number of policyholders goes to infinity, i.e. it is diversifiable. In the next proposition, we show that the unsystematic mortality risk implied by the MRT decomposition also satisfies this property. The following lemma will simplify the proof.

**Lemma 4.16.** *Let $T \in [0, T^*]$ be fixed. If $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$ and if $\left(\psi^N(t)\right)_{0\le t\le T}$ is a $\mathbb{G}$-predictable process with $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left[\psi^N(t)\right]^4\right) < \infty$, then*

$$\frac{1}{m}\int_{0+}^{T} \psi^N(t)dM^N(t) \xrightarrow[m\to\infty]{L^2} 0.$$

*Proof.* We need to show that

$$\mathrm{E}^{\mathbb{P}}\left(\left[\frac{1}{m}\int_{0+}^{T} \psi^N(t)dM^N(t) - 0\right]^2\right) = \frac{1}{m^2}\mathrm{E}^{\mathbb{P}}\left(\left[\int_{0+}^{T} \psi^N(t)dM^N(t)\right]^2\right) \xrightarrow[m\to\infty]{} 0.$$

From Andersen et al. (1997, p. 78), we know that the predictable quadratic variation of $M^N(t)$ equals $\left\langle M^N, M^N\right\rangle(t) = \int_0^t (m - N(s-))\mu(s)ds$. Since $M^N(t)$ is a martingale (Bielecki and Rutkowski, 2004, p. 153) and $\psi^N$ is assumed to be predictable, it follows (see also below) that $\int_{0+}^{T} \psi^N(t)dM^N(t)$ is a square integrable martingale, and the Itô isometry yields (for both, see Klebaner, 2005, p. 234)

$$\begin{aligned}
&\frac{1}{m^2}\mathrm{E}^{\mathbb{P}}\left(\left[\int_{0+}^{T} \psi^N(t)dM^N(t)\right]^2\right) = \frac{1}{m^2}\mathrm{E}^{\mathbb{P}}\left(\int_{0+}^{T} \left[\psi^N(t)\right]^2 \underbrace{(m - N(s-))}_{\le m}\mu(s)ds\right) \\
&\le \frac{1}{m}\mathrm{E}^{\mathbb{P}}\left(\int_{0+}^{T} \left[\psi^N(t)\right]^2 \mu(s)ds\right).
\end{aligned} \tag{4.12}$$

Since by assumption $C_1 := \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left[\psi^N(t)\right]^4\right) < \infty$ and $C_2 := \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$, the theorem of Fubini-Tonelli and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left(\int_{0+}^{T} \left[\psi^N(t)\right]^2 \mu(s)ds\right) = \int_{0+}^{T} \mathrm{E}^{\mathbb{P}}\left(\left[\psi^N(t)\right]^2 \mu(s)\right)ds \\
&\overset{\text{Cauchy-Schwarz}}{\le} \int_{0+}^{T} \sqrt{\mathrm{E}^{\mathbb{P}}\left([\psi^N(t)]^4\right)\mathrm{E}^{\mathbb{P}}\left(\mu^2(s)\right)}ds \le \int_{0+}^{T} \sqrt{C_1C_2} = T\sqrt{C_1C_2} =: C < \infty.
\end{aligned}$$

Together with (4.12) we obtain

$$0 \le \frac{1}{m^2}\mathrm{E}^{\mathbb{P}}\left(\left[\int_{0+}^{T} \psi^N(t)dM^N(t)\right]^2\right) \le \frac{1}{m}\mathrm{E}^{\mathbb{P}}\left(\int_{0+}^{T} \left[\psi^N(t)\right]^2 \mu(s)ds\right) \le \frac{1}{m}C \xrightarrow[m\to\infty]{} 0.$$

□

In order to show $L^2$-convergence, the following proposition is restricted to bounded $C_a$ and $C_{ad}$. However, convergence in probability could be shown under less restrictive assumptions.

**Proposition 4.17.** *Assume the setting and assumptions from Proposition 4.10 with resulting unsystematic mortality risks in part ii), iii) and iv) of, respectively,*

$$\begin{aligned}
R^m_{unsys,ak} &= -\int_{0+}^{t_k} \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\,\mathcal{G}_t\right)dM^N(t), \\
R^m_{unsys,a} &= -\int_{0+}^{T}\int_{t}^{T} \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)ds\,dM^N(t), \\
R^m_{unsys,ad} &= -\int_{0+}^{T}\left[\mathrm{E}^{\mathbb{P}}\left(\int_{t}^{T} C_{ad}(s)e^{\Gamma(t)-\Gamma(s)}d\Gamma(s)\middle|\,\mathcal{G}_t\right) - C_{ad}(t)\right]dM^N(t).
\end{aligned}$$

*i) If additionally* $C_{a,k} \in L^4(\mathbb{P})$ *and* $\sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$, *then* $\frac{1}{m} R^m_{unsys,ak} \xrightarrow[m\to\infty]{L^2} 0.$

*ii) If additionally* $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$ *and* $C_a$ *is bounded, then* $\frac{1}{m} R^m_{unsys,a} \xrightarrow[m\to\infty]{L^2} 0.$

*iii) If additionally* $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^4(t)\right) < \infty$ *and* $C_{ad}$ *is bounded, then* $\frac{1}{m} R^m_{unsys,ad} \xrightarrow[m\to\infty]{L^2} 0.$

The proof of Proposition 4.17 mainly checks the assumptions of Lemma 4.16 and can be found in Appendix A. While unsystematic mortality risk diversifies, Proposition 4.19 shows that the remaining risk factors also converge with the number of contracts, but in general not to zero, i.e. they are not diversifiable. This confirms their interpretation as systematic risks, in particular since the limits no longer depend on $N(t)$. In applications, if the portfolio is sufficiently large, the limits can be used as risk approximations. Again, we first show a helpful result.

**Lemma 4.18.** *If* $\zeta = (\zeta(t))_{0\le t\le T}$ *is* $\mathbb{G}$*-adapted and has continuous paths, then for* $0 \le t_k \le T \le T^*$

$$\frac{1}{m}\int_0^T \left[(m-N(t-))e^{\Gamma(t)}\mathbb{1}_{[0,t_k]} + (m-N(t_k))e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T]}\right]\zeta(t)dW(t) \xrightarrow[m\to\infty]{P} \int_0^T \zeta(t)dW(t),$$

*where* $(W(t))_{0\le t\le T^*}$ *is a one-dimensional Brownian motion.*

*Proof.* Define

$$\zeta_m(t) := \left[\frac{(m-N(t-))}{m}e^{\Gamma(t)}\mathbb{1}_{[0,t_k]}(t) + \frac{(m-N(t_k))}{m}e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T]}(t)\right]\zeta(t).$$

If $\zeta_m = (\zeta_m(t))_{0\le t\le T}$, $m \in \mathbb{N}$, are predictable processes with $\zeta_m(t) \xrightarrow[m\to\infty]{a.s.} \zeta(t)$ for all $t \in [0,T]$, and if there exists a $W$-integrable process $\alpha = (\alpha(t))_{0\le t\le T}$ such that $|\zeta_m(t)| \le \alpha(t)$ for all $m \in \mathbb{N}$, $t \in [0,T]$, then the statement of the lemma follows by the dominated convergence theorem for Itô integrals (Protter, 2005, p. 176) . Since $\zeta$ and $\mu$ are by assumption predictable, it follows that $\zeta_m$ is predictable for each $m \in \mathbb{N}$. Furthermore, since the remaining lifetimes $\tau_x^i$, $i \in \mathbb{N}$, are assumed to be conditionally i.i.d., a conditional version of Kolmogorov's strong law of large numbers (Majerek et al., 2005, p. 154) yields that $\frac{m-N(t)}{m} \xrightarrow[m\to\infty]{a.s.} e^{-\int_0^t \mu(s)ds}$. Since $\mu(t)$ is assumed to be continuous, this implies for any $t \in [0,T]$

$$\frac{m-N(t-)}{m} \xrightarrow[m\to\infty]{a.s.} e^{-\int_0^t \mu(s)ds}.$$

As a result, $\zeta_m(t) \xrightarrow[m\to\infty]{a.s.} \zeta(t)$ for all $t \in [0,T]$. Furthermore, since $\frac{m-N(t-)}{m} \le 1$ for all $t \in [0,T]$, we have

$$|\zeta_m(t)| \le \left[e^{\Gamma(t)}\mathbb{1}_{[0,t_k]}(t) + e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T]}(t)\right]|\zeta(t)| =: \alpha(t).$$

As the norm of $\zeta(t)$ is still continuous and adapted and as the same holds for $e^{\Gamma(t)}\mathbb{1}_{[0,t_k]}(t)+e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T]}(t)$, it follows that $\alpha$ is a continuous and adapted process. This implies (Klebaner, 2005, p. 98) that $\alpha$ is $W$-integrable. Summing up, all conditions of the dominated convergence theorem are satisfied, and the statement follows. □

**Proposition 4.19.** *Assume the setting and assumptions from Proposition 4.10 with resulting systematic risks in part ii), iii) and iv) of*

$$R^m_{i,\cdot} := \int_0^T \left[(m-N(t-))e^{\Gamma(t)}\mathbb{1}_{[0,t_k]}(t) + (m-N(t_k))e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T]}(t)\right]\sum_{j=1}^d \varphi_{j,\cdot}(t)\sigma^{-1}_{ji}(t)\,dM_i^W(t),$$

*where* $0 \leq T \leq T^*$*, and for the different parts*

$$\varphi_{j,\,ak}(t) = \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\big|\,\mathcal{G}_t\right) \quad \textit{(part ii)),}$$

$$\varphi_{j,\,a}(t) = \int_t^T \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(e^{-\Gamma(s)}C_a(s)\right)\big|\,\mathcal{G}_t\right) ds \quad \textit{(part iii) where } t_k = T\textit{),}$$

$$\varphi_{j,\,ad}(t) = \mathrm{E}^{\mathbb{P}}\left(D_{t,j}\left(\int_0^T C_{ad}(v)e^{\Gamma(t)-\Gamma(v)}d\Gamma(v)\right)\Big|\,\mathcal{G}_t\right) \quad \textit{(part iv) where } t_k = T\textit{).}$$

*Then it follows that for* $i = 1, \ldots, n$

$$\frac{1}{m}R^m_{i,\cdot} \xrightarrow[m\to\infty]{P} \int_0^T \sum_{j=1}^d \varphi_{j,\cdot}(t)\sigma^{-1}_{ji}(t)dM^W_i(t).$$

The proof of Proposition 4.19 is mainly based on Lemma 4.18 and can be found in Appendix A.

# 5 Numerical example

In order to demonstrate the applicability and usefulness of the MRT decomposition, we derive the equity, interest, systematic, and unsystematic mortality risk factor of a guaranteed minimum death benefit (GMDB), visualize the risk factors' distributions, and determine their risk contributions by the Euler allocation principle. GMDBs are common guarantees added to Variable Annuities (VA), which are deferred, fund-linked annuity contracts (Bauer et al., 2008).

We assume that the VA is offered against a single premium $P_0$ paid at time 0 which is fully invested in a fund $S = (S(t))_{0\leq t\leq T^*}$ modeled as a geometric Brownian motion with drift $\mu_S$ and volatility $\sigma_S$:

$$dS(t) = \mu_S S(t)dt + \sigma_S S(t)dW_S(t), \;\; S(0) > 0,$$

where $W_S = (W_S(t))_{0\leq t\leq T^*}$ denotes a $\mathbb{P}$-Brownian motion. In case the insured dies during the VA's deferment period $[0, T]$, the GMDB guarantees that the death benefit paid at the end of the year of death equals at least the single premium $P_0$ (return of premium death benefit). This means that the insurance company assumes the risk that the single premium exceeds the account value at the end of the year of death in which case it has to make up the difference. We focus on the insurer's risk from the GMDB payoff itself and assume that the insurance company charges no fee for the additional GMDB guarantee. Thus, the policyholder's account value equals $A(t) = \frac{P_0}{S_0}S(t),\ t \in [0, T]$, and if the same contract is issued to $m$ homogeneous policyholders, the total discounted loss of the insurance company amounts to

$$L_0 = \sum_{k=1}^T \left(N(t_k) - N(t_{k-1})\right) e^{-\int_0^{t_k} r(s)ds} \max\{P_0 - A(t_k), 0\}, \tag{5.1}$$

where $r(t)$ denotes the short rate, and (the death benefit is paid at end of the year of death) $t_k = k,\ k = 0, 1, \ldots, T$. We can interpret $L_0$ as the amount of money the insurance company needs at time 0 for being able to cover the GMDB liabilities given it invests its money in the bank account. Equivalently, we could assume a single upfront fee on top of $P_0$, which would not change $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ thus leading to the same MRT decomposition. In contrast, for example, a fee extracted continuously from the invested funds would change the insurer's risk $L_0$, so that effects from the payoff and the charged fee would overlap.

The short rate $r = (r(t))_{0\leq t\leq T^*}$ is assumed to follow a Cox-Ingersoll-Ross (CIR) process

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r\sqrt{r(t)}dW_r(t), \;\; r(0) > 0,$$

where $\kappa$, $\theta$, and $\sigma_r$ denote the parameters for mean reversion speed, mean reversion level and volatility, respectively, and $W_r = (W_r(t))_{0\leq t\leq T^*}$ is a $\mathbb{P}$-Brownian motion. Since $r$ is an affine process, it follows that (Björk, 2005, Proposition 22.2, p. 331)

$$\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T r(s)ds}\Big|\,\mathcal{G}_t\right) = e^{\alpha_r(t,T)-\beta_r(t,T)r(t)},\ T\in[t,T^*],$$

where analogously to the risk-neutral case (Brigo and Mercurio, 2006, p. 66), $\alpha_r$ and $\beta_r$ can be derived as $\alpha_r(t,T) = \frac{2\kappa\theta}{\sigma_r^2}\log\left(\frac{2he^{(\kappa+h)\frac{T-t}{2}}}{2h+(\kappa+h)(e^{h(T-t)}-1)}\right)$, $\beta_r(t,T) = \frac{2(e^{h(T-t)}-1)}{2h+(\kappa+h)(e^{h(T-t)}-1)}$, $h = \sqrt{\kappa^2+2\sigma_r^2}$.

As proposed by Dahl and Møller (2006), we assume that under $\mathbb{P}$ the mortality intensity process $\mu = (\mu(t))_{0\leq t\leq T^*}$ follows a time-inhomogeneous CIR process

$$d\mu(t,x) = (\gamma(t,x)-\delta(t,x)\mu(t,x))dt + \sigma_\mu(t,x)\sqrt{\mu(t,x)}dW_\mu(t),\ \ \mu(0,x) = \mu^0(x),$$

where $x$ denotes the dependency on the age at time 0, $W_\mu = (W_\mu(t))_{0\leq t\leq T^*}$ is a $\mathbb{P}$-Brownian motion, the initial mortality intensities $\mu^0(x+t) = a + bc^{x+t}$ are assumed to follow the Gompertz-Makeham mortality law, and

$$\gamma(t,x) = \frac{1}{2}\hat{\sigma}^2\mu^0(x+t),\ \ \delta(t,x) = \hat{\delta} - \frac{\frac{d}{dt}\mu^0(x+t)}{\mu^0(x+t)},\ \ \sigma_\mu(t,x) = \hat{\sigma}\sqrt{\mu^0(x+t)},$$

for some deterministic parameters $a, b, c, \hat{\delta}$ and $\hat{\sigma}$. The specified mortality intensity process is again an affine process which implies $\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T \mu(s,x)ds}\Big|\,\mathcal{G}_t\right) = e^{\alpha_\mu(t,T,x)-\beta_\mu(t,T,x)\mu(t,x)}$, $T\in[t,T^*]$, where $\alpha_\mu$ and $\beta_\mu$ satisfy the ordinary differential equations specified in Proposition 3.1 of Dahl and Møller (2006, p. 197) and cannot be determined analytically. In what follows, we only consider a single cohort, i.e. we fix the initial age $x$, and define $\mu(t) := \mu(t,x)$.

Since we assume that $W_S$, $W_r$ and $W_\mu$ are independent one-dimensional Brownian motions, the volatility function of the process $X := (S, r, \mu)^\top$ equals

$$\sigma(t,x) = \begin{pmatrix} \sigma_S x_1 & 0 & 0 \\ 0 & \sigma_r\sqrt{x_2} & 0 \\ 0 & 0 & \sigma_\mu(t,x)\sqrt{x_3} \end{pmatrix}.$$

Thus, if $X$ remains positive, it follows that $\det\sigma(t,x)\neq 0$ for all $t\in[0,T^*]$ and for all values $x$ the process $X(t)$, $t\in[0,T^*]$, assumes. Recall that the geometric Brownian motion remains positive by definition, and the same holds for the CIR processes $r$ and $\mu$ given the Feller conditions $2\kappa\theta\geq\sigma_r^2$ and $2\gamma(t,x)\geq(\sigma_\mu(t,x))^2$ (Dahl and Møller, 2006, p. 197), respectively, are satisfied. By the definition of $\gamma$ and $\sigma_\mu$, the latter is here always satisfied.

For deriving the MRT decomposition of $L_0$ defined in (5.1), first note that $L_0$ can be rewritten as

$$\begin{aligned} L_0 = &\sum_{k=1}^{T}(m-N(t_{k-1}))\,e^{-\int_0^{t_k} r(s)ds}\max\{P_0-A(t_k),0\} \\ &-\sum_{k=1}^{T}(m-N(t_k))\,e^{-\int_0^{t_k} r(s)ds}\max\{P_0-A(t_k),0\}, \end{aligned} \tag{5.2}$$

i.e. it is a sum of survival benefits. Since $X$ is a Markov process, we can define the functions

$$f_k^{A1}(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^{t_{k-1}}\mu(s,X(s))ds}e^{-\int_t^{t_k} r(s)ds}\max\{P_0-A(t_k),0\}\Big|\,X(t)=x\right),\ \ 0\leq t\leq t_{k-1},$$

$$f_k^{B1}(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^{t_k} r(s)ds}\max\{P_0-A(t_k),0\}\Big|\,X(t)=x\right),\ \ 0\leq t\leq t_k,$$

$$f_k^{A2}(t,x) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^{t_k}[r(s)+\mu(s,X(s))]ds}\max\{P_0-A(t_k),0\}\Big|\,X(t)=x\right),\ \ 0\leq t\leq t_k,$$

which can be simplified by using the independence of $S,\ r$, and $\mu$, as well as exploiting the normal distribution of $S$, and the affine property of $r$ and $\mu$. This immediately shows that all three functions are sufficiently smooth, so that we can apply part a) of Proposition 4.13 ii) to derive the MRT decompositions of the summands in the first line of (5.2) (using $f_k^{A1}$ and $f_k^{B1}$), and accordingly part b) of Proposition 4.13 ii) to derive the MRT decompositions of the summands in the second line of (5.2) (using $f_k^{A2}$). Note that all conditions of Proposition 4.13 ii) are actually satisfied. Altogether, we obtain for the MRT decomposition of $L_0$ that

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \sum_{i=1}^{n+1} R_i,$$

where the systematic risk factors implied by $X_i,\ i = 1, 2, 3$, where $X = (S, r, \mu)$, equal

$$\begin{aligned} R_i := \sum_{k=1}^{T} \Bigg( & \int_0^{t_{k-1}} (m - N(t-)) e^{-\int_0^t r(s)ds} \frac{\partial}{\partial x_i} f_k^{A1}(t, X(t)) dM_i^W(t) \\ & + \int_{t_{k-1}}^{t_k} (m - N(t_{k-1})) e^{-\int_0^t r(s)ds} \frac{\partial}{\partial x_i} f_k^{B1}(t, X(t)) dM_i^W(t) \Bigg) \\ & - \sum_{k=1}^{T} \int_0^{t_k} (m - N(t-)) e^{-\int_0^t r(s)ds} \frac{\partial}{\partial x_i} f_k^{A2}(t, X(t)) dM_i^W(t), \end{aligned}$$

and the unsystematic mortality risk factor is given by

$$R_{n+1} := -\sum_{k=1}^{T} \int_{0+}^{t_{k-1}} e^{-\int_0^t r(s)ds} f_k^{A1}(t, X(t)) dM^N(t) + \sum_{k=1}^{T} \int_{0+}^{t_k} e^{-\int_0^t r(s)ds} f_k^{A2}(t, X(t)) dM^N(t).$$

Note that the independence of $W_S,\ W_r$ and $W_\mu$ was not a necessary assumption for applying Proposition 4.13 (as long as $\det \sigma(t, x) \neq 0$), but significantly simplified the verification of the required smoothness properties, and prevents the need of nested simulations in the numerical calculations.

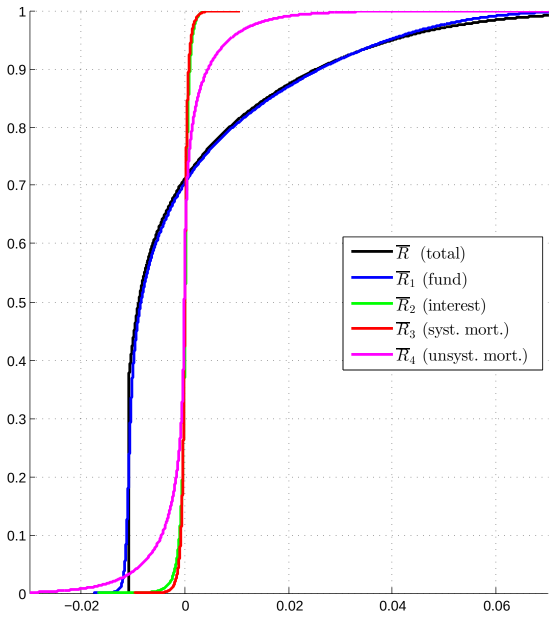
For the numerical calculations, we assume $m = 100$ GMDB contracts with maturity $T = 15$ and single premium $P_0 = 100{,}000$. All policyholders are assumed to be of age $x = 50$ at time 0. We perform $N = 100{,}000$ simulations for determining the distributions of $R := L_0 - \mathrm{E}^{\mathbb{P}}(L_0), R_1, R_2, R_3$, and $R_4$. For projecting the risk drivers $r$ and $\mu$ as well as for approximating the stochastic integrals, we use the Euler scheme with $n = 100$ time steps per year. The number of survivors in the portfolio are projected by means of the binomial distribution conditioned on the mortality intensities. The ODEs implied by the mortality model are solved numerically. With respect to the mortality model, we adopt the parameter values for year 2003, case II, males, from Tables 1 to 3 in Dahl and Møller (2006, p. 211): $a = 0.000134,\ b = 0.0000353,\ c = 1.1020,\ \hat{\hat{\delta}} = 0.008$, and $\hat{\sigma} = 0.02$. For the interest model, we assume $\kappa = 0.2,\ \theta = 0.025,\ \sigma_r = 0.075$ and $r(0) = 0.0029$. Thus, the Feller condition $2\kappa\theta = 0.01 > 0.0056 = \sigma_r^2$ is satisfied. The parameters of the geometric Brownian motion are $\mu_S = 0.06$ and $\sigma_S = 0.22$.

We focus on the distributions scaled by the number of policyholders in the portfolio and the single premium, i.e. we consider $\overline{R} := \frac{1}{mP_0} R,\ \overline{R}_i := \frac{1}{mP_0} R_i,\ i = 1, \ldots, 4$. The resulting empirical distribution functions of the total risk $\overline{R}$, the fund risk $\overline{R}_1$, the interest risk $\overline{R}_2$, the systematic mortality risk $\overline{R}_3$, and the unsystematic mortality risk $\overline{R}_4$ are shown in Figure 1(a). The distribution functions directly imply that the fund is the most relevant risk driver since its risk factor exhibits almost the same distribution as the total risk. This appears to be reasonable since it only depends on the fund value whether the GMDB guarantee is in the money or not in case of death. For $m = 100$ contracts, the randomness of the number of deaths within $[0, T]$, which trigger a possibly positive payoff, also seems to be rather high. This is why the range of the unsystematic mortality risk factor is rather wide compared to the ranges of the interest risk and the systematic mortality risk factor. In contrast, the latter two risk factors seem to be negligible since their realizations hardly deviate from zero. In particular, the randomness of the stochastic mortality intensity has almost no influence on the total risk which can explained by its low volatility parameter. We also see that the distribution function of the fund risk factor is right-skewed while the distribution functions of all other risk factors are more or less symmetric. In Figure 1(b) we plotted the histogram values of the interest and mortality risk factors using a bin size of $1e$-4. Obviously, the histogram of the interest risk is slightly shifted to the right compared to the other two histograms, i.e. more weight is on the risky side. Furthermore, the tails of the interest risk are heavier than the tails of the systematic mortality risk, but lighter than the tails of the systematic mortality risk.

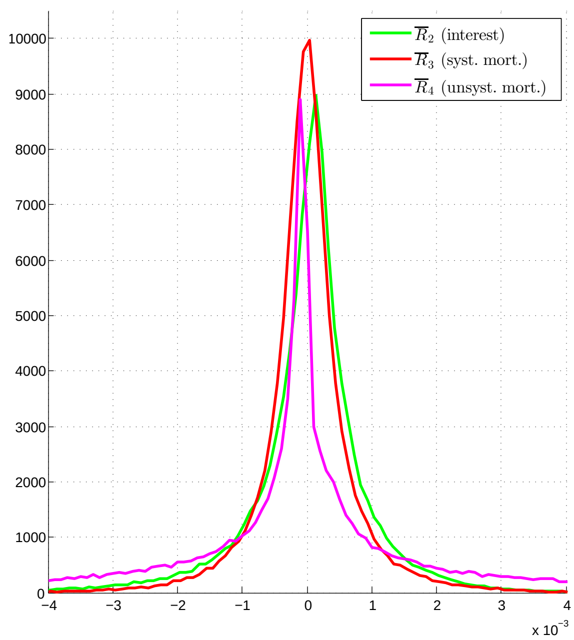
Clearly, since the stochastic differential equations of the risk drivers $r$ and $\mu$, as well as the stochastic integrals of the risk factors are numerically approximated, the distributions and histogram values are only approximations. In particular, comparing the left-hand side $\overline{R}$ and the right-hand side $\overline{R}_1 + \overline{R}_2 + \overline{R}_3 + \overline{R}_4$ of the decomposition, we obtain

- for the absolute error: $\mathrm{E}^{\mathbb{P}}\left(\left|\overline{R} - \left(\overline{R}_1 + \overline{R}_2 + \overline{R}_3 + \overline{R}_4\right)\right|\right) = 3.7173e\text{-}04$,
- for the weak error: $\left|\mathrm{E}^{\mathbb{P}}\left(\overline{R}\right) - \mathrm{E}^{\mathbb{P}}\left(\overline{R}_1 + \overline{R}_2 + \overline{R}_3 + \overline{R}_4\right)\right| = 6.4665e\text{-}06$.

Clearly, the distribution of $\overline{R}$, which is the distribution of an option, is not completely absolutely continuous, since $\overline{R}$ equals $-\mathrm{E}^{\mathbb{P}}(L_0)$ with positive probability. The same actually also holds for the risk factors $\overline{R}_i$, $i = 1, \ldots, 4$, but as a result of the discretization of the risk factor integrals, the approximated distributions seem to be absolutely continuous. For solving this problem, more advanced numerical procedures would be necessary. However, this goes beyond the scope of this paper.




(a) Cumulative distribution functions of the standardized risk factors $\overline{R}$, $\overline{R}_1$, $\overline{R}_2$, $\overline{R}_3$, and $\overline{R}_4$.




(b) Plotted histogram values of the standardized risk factors $\overline{R}_2$, $\overline{R}_3$, and $\overline{R}_4$.

Figure 1: GMDB portfolio with $m = 100$ contracts.

In order to quantify the total risk we apply three different risk measures:

- Standard deviation: $\rho(\overline{R}) = \sqrt{\mathrm{Var}\left(\overline{R}\right)}$,

- Value-at-Risk at the 99% level: $\rho(\overline{R}) = \inf\left\{x \in \mathbb{R} : \mathbb{P}\left(\overline{R} \leq x\right) \geq 0.99\right\}$,
- Tail-Value-at-Risk: $\rho(\overline{R}) = \mathrm{E}^{\mathbb{P}}\left(\overline{R} \,\middle|\, \overline{R} \geq \mathrm{VaR}_{0.99}\left(\overline{R}\right)\right)$,

The total risk $\rho(\overline{R})$ is allocated to the four sources of risk by means of the Euler principle (for an extensive discussion of allocation principles, see Bauer and Zanjani, 2013). In fact, since the chosen risk measures are (positively) homogeneous, Euler's homogeneous function theorem yields

$$\rho(\overline{R}) = \sum_{i=1}^{4} \left.\frac{\partial \rho\left(a_1\overline{R}_1 + a_2\overline{R}_2 + a_3\overline{R}_3 + a_4\overline{R}_4\right)}{\partial a_i}\right|_{a_1=a_2=a_3=a_4=1},$$

and each summand can be interpreted as the risk contribution of the respective risk factor. We have assumed that $\rho\left(a_1\overline{R}_1 + a_2\overline{R}_2 + a_3\overline{R}_3 + a_4\overline{R}_4\right)$ is differentiable in each $a_i,\ i = 1, \ldots, 4$. For the standard deviation as risk measure, this can be easily shown and the Euler allocation results in the well-known covariance principle (McNeil et al., 2005, p. 258). For the other two risk measures, differentiability is not clear, but a numerical approximation of the respective functions suggests sufficient smoothness. Thus, in what follows we also assume differentiability for value-at-risk and tail-value-at-risk and numerically approximate the respective risk contributions via

$$\left.\frac{\partial \rho\left(a_1\overline{R}_1 + a_2\overline{R}_2 + a_3\overline{R}_3 + a_4\overline{R}_4\right)}{\partial a_i}\right|_{a_1=a_2=a_3=a_4=1} \approx \frac{\rho(\overline{R} + h\overline{R}_i) - \rho(\overline{R} - h\overline{R}_i)}{2h},$$

$i = 1, \ldots, 4$, where we choose $h = 0.01$. In Table 2 we see for each risk measure the quantified total risk $\rho(\overline{R})$ and the allocated risk contributions in the first line (rounded to four decimals), and all values from the first line as a percentage of the sum of the four individual risk contributions in the second line. As a result of the numerical approximations, the allocated values do not perfectly add up to the total risk $\rho(\overline{R})$, in particular with the value-at-risk allocation.

| | | $\overline{R}$ (total) | $\overline{R}_1$ (fund) | $\overline{R}_2$ (interest) | $\overline{R}_3$ (syst. mort.) | $\overline{R}_4$ (unsyst. mort.) |
|---|---|---|---|---|---|---|
| Std. dev. | abs. | 0.0179 | 0.0160 | 0.0001 | 0.0000 | 0.0018 |
| | perc. | 100.0% | 89.3% | 0.3% | 0.2% | 10.1% |
| $\mathrm{VaR}_{0.99}$ | abs. | 0.0675 | 0.0553 | 0.0005 | 0.0002 | 0.0117 |
| | perc. | 99.6% | 81.7% | 0.8% | 0.2% | 17.3% |
| $\mathrm{TVaR}_{0.99}$ | abs. | 0.0813 | 0.0592 | 0.0008 | 0.0004 | 0.0208 |
| | perc. | 100.0% | 72.9% | 1.0% | 0.5% | 25.6% |

Table 2: Euler risk contributions for a GMDB portfolio with $m = 100$ contracts in absolute terms (abs.), and relative to the sum of the four individual risk contributions (perc.).

The allocated risk contributions confirm our observations from the empirical distributions and the histogram values. With all three risk measures, the fund is responsible for at least 70% of the total risk. The risk contributions of interest and systematic mortality are at about the same low level. The unsystematic mortality risk implies a risk contribution of 10.1% with the standard deviation, 17.3% with the value-at-risk, and 25.6% with the tail-value-at-risk Euler allocation. Obviously, the unsystematic mortality risk becomes more relevant in the tail of the total risk, which makes sense keeping in mind that high losses can only occur if the policyholder actually dies.

Increasing the number of policies from $m = 100$ to $m = 10{,}000$ results in the distributions from Figure 2. Confirming our theoretical convergence results from Section 4.3, the increased portfolio size immensely reduces the unsystematic mortality risk per contract, whereas the systematic risks per contract remain more or less unaffected. Since the total risk per contract equals the sum of the individual risks, it also decreases.

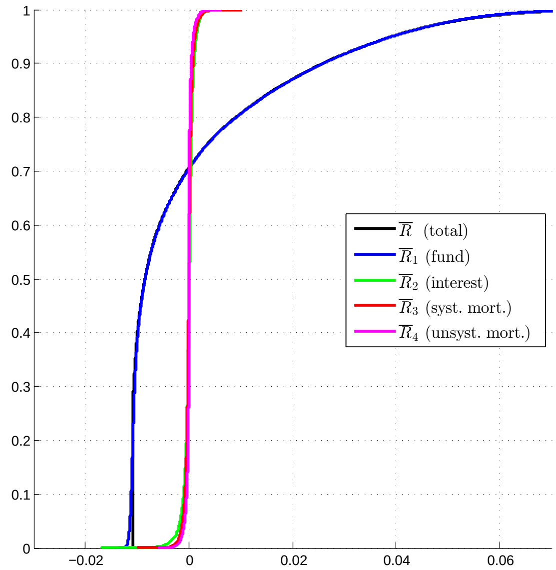



Figure 2: Cumulative distribution functions of the standardized risk factors $\overline{R}$, $\overline{R}_1$, $\overline{R}_2$, $\overline{R}_3$, and $\overline{R}_4$ for a GMDB portfolio with $m = 10{,}000$ contracts.

# 6 Conclusion

The present paper provides a profound analysis of risk decomposition methods which allocate the randomness of life insurance liabilities to risk factors associated with different sources of risk. For evaluating the usefulness of different approaches, we first introduced a list of properties we posit a meaningful risk decomposition should satisfy. Then we proposed a novel decomposition method, labeled MRT decomposition, and showed that it satisfies all of these desirable properties. In contrast, it turned out that for each of the decomposition approaches proposed in literature so far at least one of the properties fails to hold. As opposed to most literature, we explicitly focus on the decomposition of life insurance liabilities into risk factors, which are again random variables, and not on quantifying the risk contributions with the help of risk measures. A decomposition into random risk factors allows not only the application of the risk measure of choice in later quantitative comparisons, but also enables an analysis of the risk structure with respect to dependencies or distributional characteristics of the risk factors. Furthermore, it is clear that we first need to understand the properties of the decomposition method underlying a risk quantification before a reliable interpretation of the results can become possible.

Our alternative decomposition method is mainly based on the martingale representation theorem, thus labeled MRT decomposition, and decomposes life insurance liabilities into Itô integrals with respect to the compensated sources of risk. The considered life insurance setting is rather general with an arbitrary (finite) insurance portfolio modeled by a counting process, and an insurance payoff entailing discrete as well as continuous survival and death benefits. The (systematic) sources of risk are assumed to be driven by a finite-dimensional Brownian motion. We showed that, under certain assumptions, the MRT decomposition exists and is unique, which still holds if the driving one-dimensional Brownian motions are correlated. We were able to first isolate the influence of the unsystematic risk, so that explicit formulas for the MRT decomposition could be derived by means of the Clark-Ocone formula from Malliavin calculus in the general case and by Itô's lemma for diffusion processes in the Markov case.

We came back to the desirable properties derived in Section 2.1 and proved that the MRT decomposition satisfies all meaningful risk decomposition properties. We also showed that the unsystematic mortality risk as specified by the MRT decomposition is diversifiable, i.e. it vanishes as the portfolio increases, whereas the systematic risk factors approach a non-zero limit. This observation agrees with common opinion, and confirms the interpretation of the integrals. Furthermore, if the portfolio is sufficiently large, the limits can provide useful risk approximations in applications.

In Section 2.2 the paper reviews several decomposition methods from the actuarial literature. By means of simple examples, we detected serious drawbacks within all approaches, so that none of them satisfies all meaningful risk decomposition properties. Above all, the well-known and widely used variance decomposition is based on a risk decomposition method which is not unique, in the sense that the order of conditioning matters. As a result, in some examples the method yields doubtable results. The Hoeffding decomposition eliminates the drawback of the stochastic variance decomposition. However, it introduces co-movement factors which cannot be allocated to a specific source of risk, but sometimes capture the total randomness. Thus, the method's usefulness for decomposing risk is not always given. Finally, the results from the Taylor expansion and the Solvency II approach are more than sensitive with respect to the required problem-specific choices.

In Section 5 we explicitly calculated the MRT decomposition of a GMDB policy with return-of-premium guarantee. Interest and mortality rates were assumed to follow affine processes and the Variable Annuity account was modeled as a geometric Brownian motion. Thus, we analyzed the influence of four sources of risk: equity, interest rate, aggregate mortality, and actual deaths observed in the portfolio of insured. We illustrated the distributions of the total risk and all four risk factors and quantified the respective risk contributions by the Euler allocation principle. Our calculations show that for an unhedged exposure, equity risk is by far the most dominant risk, particularly when considering moderately sized insurance portfolios. Furthermore, unsystematic mortality risk immensely reduces when increasing the number of policyholders from $100$ to $10{,}000$ confirming the theoretical result. In particular, this example demonstrates the applicability and usefulness of the MRT decomposition.

In summary, the present paper shows that the appropriateness of the application of currently prevailing decomposition methods such as the stochastic foundation of the variance decomposition approach can be questioned. Instead, we propose an alternative method, the so-called MRT decomposition, which seems to be promising. For future research, it would be desirable to transfer the MRT decomposition to a more general Lévy setting, and to find approximations of the risk factors to reduce computational efforts. An application of the method to more advanced insurance guarantees such as guaranteed annuity options as well as a more detailed analysis of the decomposition results would be interesting. Finally, since our life insurance setting is closely related to credit risk settings a transfer of the MRT decomposition to this field of research could be worthwhile.

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# Appendix

# A Proofs

*Proof of Lemma 3.2.* i) Assuming that the drift vector $\theta$ is $\mathbb{G}$-adapted with continuous paths, it follows that $A_i^W$ is a predictable finite variation process. Since $M_i^W$ is a local martingale and $X_i(t) = X_i(0) + M_i^W(t) + A_i^W(t)$ for all $t \in [0, T^*]$, $A_i^W$ is a compensator of $X_i$. The uniqueness follows by Theorem 34 in Protter (2005, p. 130).

ii) As a result of the assumptions, $A^N$ is a predictable finite variation process and $M^N$ is a martingale (for the latter, cf. Bielecki and Rutkowski, 2004, p. 153). Thus, $A^N$ is a compensator of $N$ and the uniqueness again follows by Theorem 34 in Protter (2005, p. 130).

□

*Proof of Lemma 4.4.* Since $U = (U(t))_{0\leq t\leq T^*}$ with $U(t) := \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(T)}F\middle|\mathcal{G}_t\right)$ is a $\mathbb{G}$-martingale, it follows by the martingale representation theorem that there exist predictable processes $\varphi_1,\ldots,\varphi_d$ such that (4.4) holds. We first show the lemma for a single policyholder with remaining lifetime $\tau_x^i$, i.e. $m=1$ and $\mathbb{F}=\mathbb{G}\vee\mathbb{I}^i$ for any arbitrary, but fixed $i\in\{1,\ldots,m\}$. In contrast to the proof of Proposition 5.2.2 in Bielecki and Rutkowski (2004, pp. 159), we apply the integration by parts formula here to the product $\tilde{L}_i(t)U(t)$ instead of $L_i(t)U(t)$, where $L_i(t) := \mathbb{1}_{\{\tau_x^i>t\}}e^{\Gamma(t)}$ and $\tilde{L}_i(t) := \mathrm{E}^{\mathbb{P}}\left(L_i(T)\middle|\mathcal{F}_t\right)$ for any $i\in\{1,\ldots,m\}$. This will lead to corrected integrands of the $dW_i$-terms. Since $L_i(t)$ is an $\mathbb{F}$-martingale (Bielecki and Rutkowski, 2004, p. 152), it follows that $\tilde{L}_i(t) = L_i(t)$ for $t\leq T$, and $\tilde{L}_i(t) = L_i(T)$ for $t\geq T$. Furthermore, it holds $U(T^*) = e^{-\Gamma(T)}F$ which implies $Z_i := \mathbb{1}_{\{\tau_x^i>T\}}F = \tilde{L}_i(T^*)U(T^*)$. Thus, applying the Itô integration by parts formula (Protter, 2005, p. 68) to the product $\tilde{L}_i(t)U(t)$ and considering the continuity of $U(t)$ yields

$$\begin{aligned} Z_i &= \tilde{L}_i(0)U(0) + \int_0^{T^*}\tilde{L}_i(t-)dU(t) + \int_{0+}^{T^*}U(t)d\tilde{L}_i(t) + [\tilde{L}_i,U]_{T^*} \\ &= L_i(0)U(0) + \int_0^{T^*}\left[L_i(t-)\mathbb{1}_{[0,T]}(t) + L_i(T)\mathbb{1}_{(T,T^*]}\right]dU(t) + \int_{0+}^{T}U(t)dL_i(t) + [L_i,U]_T, \end{aligned} \tag{A.1}$$

where the second equality follows from the definition of $\tilde{L}_i$. Using $\mathbb{1}_{\{\tau_x^i>0\}}=1$ a.s. (which follows from the assumptions on $\mu$), (3.1), and the $\mathcal{G}_{T^*}$-measurability of $F$, we have that

$$L_i(0)U(0) \overset{\text{a.s.}}{=} \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(T)}F\right) = \mathrm{E}^{\mathbb{P}}\left(\mathrm{E}^{\mathbb{P}}\left(\mathbb{1}_{\{\tau_x^i>T\}}\middle|\mathcal{G}_{T^*}\right)F\right) = \mathrm{E}^{\mathbb{P}}\left(\mathbb{1}_{\{\tau_x^i>T\}}F\right) = \mathrm{E}^{\mathbb{P}}(Z_i).$$

Also note that

$$M_i^N(t) := \mathbb{1}_{\{\tau_x^i\leq t\}} - \int_0^t \mathbb{1}_{\{\tau_x^i>s-\}}\mu(s)ds = \mathbb{1}_{\{\tau_x^i\leq t\}} - \int_0^{t\wedge\tau_x^i}\mu(s)ds.$$

Thus, since the $\mathbb{G}$-adapted cumulative mortality intensity $\Gamma$ of $\tau_x^i$ is continuous and increasing, Proposition 5.1.3 (i) from Bielecki and Rutkowski (2004, p. 153) implies that

$$dL_i(t) = -L_i(t-)dM_i^N(t).$$

Plugging in the definitions of $L_i$ and $M_i^N$, this can be further transformed into

$$dL_i(t) = -e^{\Gamma(t)}\left(\mathbb{1}_{\{\tau_x^i>t-\}}d\mathbb{1}_{\{\tau_x^i\leq t\}} - \mathbb{1}_{\{\tau_x^i>t-\}}\mathbb{1}_{\{\tau_x^i>t\}}\mu(t)dt\right) = -e^{\Gamma(t)}dM_i^N(t).$$

Moreover, $[L_i,U]_t = 0$ for every $t\in[0,T^*]$ (Bielecki and Rutkowski, 2004, p. 160). Additionally using the martingale representation of $U(t)$, equation (A.1) becomes

$$Z_i = \mathrm{E}^{\mathbb{P}}(Z_i) + \sum_{j=1}^d\int_0^{T^*}\left[L_i(t-)\mathbb{1}_{[0,T]}(t) + L_i(T)\mathbb{1}_{(T,T^*]}\right]\varphi_j(t)dW_j(t) - \int_{0+}^T U(t)e^{\Gamma(t)}dM_i^N(t).$$

Together with the continuity and adaptedness of $\mu$, this proves the statement of the proposition for any single policyholder. In the portfolio case, where $\mathbb{F}=\mathbb{G}\vee\bigvee_{i=1}^m\mathbb{I}^i$, the conditional independence of the $\tau_x^i$'s implies that $\mathrm{E}^{\mathbb{P}}\left(Z_i\middle|\mathcal{F}_t\right) = \mathrm{E}^{\mathbb{P}}\left(Z_i\middle|\mathcal{G}_t\vee\mathcal{I}_t^i\right)$. Thus, additionally using the conditionally identical distribution of $\tau_x^i$, $i=1,\ldots,m$, the proposition follows for the entire portfolio from applying the previous part of the proof to each summand of $Z=\sum_{i=1}^m\mathbb{1}_{\{\tau_x^i>T\}}F$ separately and adding the respective decompositions. □

*Proof of Lemma 4.6.* Note that by the martingale representation theorem, there exist predictable processes $\varphi_1, \dots, \varphi_d$ such that (4.6) holds. We first show the statement for a single policyholder with remaining life $\tau_x^i$, i.e. $m = 1$ and $\mathbb{F} = \mathbb{G} \vee \mathbb{I}^i$ for an arbitrary, but fixed $i \in \{1, \dots, m\}$. Since $F$ is assumed to be $\mathbb{G}$-predictable with $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]} |F(t)|\right) < \infty$, it follows from Proposition 5.1.2 in Bielecki and Rutkowski (2004, p. 149) that

$$\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left(\int_0^T \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv \middle| \mathcal{F}_t\right)\\
&= \int_0^t \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv + \mathrm{E}^{\mathbb{P}}\left(\int_t^T \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv \middle| \mathcal{F}_t\right)\\
&= \int_0^t \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv + L_i(t)\,\mathrm{E}^{\mathbb{P}}\left(\int_t^T e^{-\Gamma(v)} F(v)dv \middle| \mathcal{G}_t\right)\\
&= \int_0^t \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv - L_i(t)\int_0^t e^{-\Gamma(v)} F(v)dv + L_i(t)\,\mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)} F(v)dv \middle| \mathcal{G}_t\right), \qquad \text{(A.2)}
\end{aligned}$$

where $L_i(t) := \mathbb{1}_{\{\tau_x^i>t\}} e^{\Gamma(t)}$. Note that Proposition 5.1.2 in Bielecki and Rutkowski (2004) actually requires $\int_0^T F(s)ds$ to be bounded. However, via dominated convergence it can be shown that the result still holds if F satisfies $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]} |F(t)|\right) < \infty$. Biagini et al. (2012, p. 22) already pointed out a possible relaxation to $\mathrm{E}^{\mathbb{P}}\left(\sup_{t\in[0,T]} |F(t)|^2\right) < \infty$.

As in the proof of Lemma 4.4, it follows by applying integration by parts to the last two addends of (A.2) that

$$L_i(t)\int_0^t e^{-\Gamma(v)} F(v)dv = \int_0^t \mathbb{1}_{\{\tau_x^i>s-\}} F(s)ds - \int_0^t \left(\int_0^s e^{-\Gamma(v)} F(v)dv\right) e^{\Gamma(s)} dM_i^N(s),$$

and

$$\begin{aligned}
&L_i(t)\,\mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)} F(v)dv \middle| \mathcal{G}_t\right)\\
&= \mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)} F(v)dv\right) + \sum_{i=1}^d \int_0^t \mathbb{1}_{\{\tau_x^i>s-\}} e^{\Gamma(s)} \varphi_i(s) dW_i(s)\\
&\quad - \int_0^t \mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)} F(v)dv \middle| \mathcal{G}_s\right) e^{\Gamma(s)} dM_i^N(s)
\end{aligned}$$

where $M_i^N(t) := \mathbb{1}_{\{\tau_x^i\le t\}} - \int_0^t \mathbb{1}_{\{\tau_x^i>s-\}} \mu(s)ds$. Summing up the representations of all summands from (A.2) and using the $\mathcal{F}_T$-measurability of $\int_0^T \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv$, we obtain

$$\begin{aligned}
\int_0^T \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv = {}& \mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)} F(v)dv\right) + \sum_{i=1}^d \int_0^T \mathbb{1}_{\{\tau_x^i>t-\}} e^{\Gamma(t)} \varphi_i(t) dW_i(t)\\
& - \int_{0+}^T \mathrm{E}^{\mathbb{P}}\left(\int_t^T e^{\Gamma(t)-\Gamma(v)} F(v)dv \middle| \mathcal{G}_t\right) dM_i^N(t).
\end{aligned}$$

Additionally, the theorem of Fubini-Tonelli and the construction of $\tau_x^i$ imply that

$$\mathrm{E}^{\mathbb{P}}\left(\int_0^T e^{-\Gamma(v)} F(v)dv\right) = \mathrm{E}^{\mathbb{P}}\left(\int_0^T \mathbb{1}_{\{\tau_x^i>v\}} F(v)dv\right).$$

By the conditional independence assumption on $\tau_x^i,\ i = 1, \dots, m,$ we have in the portfolio case with $\mathbb{F} = \mathbb{G} \vee \bigvee_{i=1}^m \mathbb{I}^i$ that $\mathrm{E}^{\mathbb{P}}\left( \int_0^T \mathbb{1}_{\{\tau_x^i > v\}} F(v)dv \middle| \mathcal{F}_t \right) = \mathrm{E}^{\mathbb{P}}\left( \int_0^T \mathbb{1}_{\{\tau_x^i > v\}} F(v)dv \middle| \mathcal{G}_t \vee \mathcal{I}_t^i \right)$. Thus, the statement for the portfolio directly follows by applying the obtained equation to each summand $\int_0^T \mathbb{1}_{\{\tau_x^i > v\}} F(v)dv,\ i = 1, \dots, m,$ separately and adding the respective decompositions. □

*Proof of Lemma 4.8.* Note that by the martingale representation theorem, there exist predictable processes $\varphi_1, \dots, \varphi_d$ such that (4.8) holds. Since $F$ is continuous, it follows from the definition of Lebesgue integrals that

$$Z = \int_0^T F(v)dN(v) = \sum_{i=1}^m \mathbb{1}_{\{\tau_x^i \le T\}} F(\tau_x^i). \tag{A.3}$$

We first show the statement for a single policyholder with remaining life $\tau_x^i$, i.e. $m = 1$ and $\mathbb{F} = \mathbb{G} \vee \mathbb{I}^i$ for an arbitrary, but fixed $i \in \{1, \dots, m\}$. Observe that

$$\mathrm{E}^{\mathbb{P}}\left( \mathbb{1}_{\{\tau_x^i \le T\}} F(\tau_x^i) \middle| \mathcal{F}_t \right) = \mathrm{E}^{\mathbb{P}}\left( \mathbb{1}_{\{t < \tau_x^i \le T\}} F(\tau_x^i) \middle| \mathcal{F}_t \right) + \mathbb{1}_{\{\tau_x^i \le t\}} F(\tau_x^i), \tag{A.4}$$

since $\mathbb{1}_{\{\tau_x^i \le t\}} F(\tau_x^i)$ is $\mathcal{F}_t$-measurable. Since $F$ is assumed to be $\mathbb{G}$-predictable with $\mathrm{E}^{\mathbb{P}}\left( \sup_{t \in [0,T]} |F(t)| \right) < \infty$, it follows from Corollary 5.1.3 in Bielecki and Rutkowski (2004, p. 148) that

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left( \mathbb{1}_{\{t < \tau_x^i \le T\}} F(\tau_x^i) \middle| \mathcal{F}_t \right) &= \mathbb{1}_{\{\tau_x^i > t\}} \mathrm{E}^{\mathbb{P}}\left( \int_t^T F(v) e^{\Gamma(t) - \Gamma(v)} d\Gamma(v) \middle| \mathcal{G}_t \right) \\
&= L_i(t) \mathrm{E}^{\mathbb{P}}\left( \int_0^T F(v) e^{-\Gamma(v)} d\Gamma(v) \middle| \mathcal{G}_t \right) - L_i(t) \int_0^t F(v) e^{-\Gamma(v)} d\Gamma(v),
\end{aligned}$$

where $L_i(t) := \mathbb{1}_{\{\tau_x^i > t\}} e^{\Gamma(t)}$. Again, Proposition 5.1.1 and thus Corollary 5.1.3 in Bielecki and Rutkowski (2004, p. 148) actually require $F$ to be bounded. However, via dominated convergence it can be shown that both results generalize to non-bounded processes F that satisfy $\mathrm{E}^{\mathbb{P}}\left( \sup_{t \in [0,T]} |F(t)| \right) < \infty$. Biagini et al. (2012, p. 19) already pointed out a possible relaxation to $\mathrm{E}^{\mathbb{P}}\left( \sup_{t \in [0,T]} |F(t)|^2 \right) < \infty$. As in the proof of Lemma 4.4, it then follows by applying integration by parts to both addends that

$$\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left( \mathbb{1}_{\{t < \tau_x^i \le T\}} F(\tau_x^i) \middle| \mathcal{F}_t \right) \\
&= \mathrm{E}^{\mathbb{P}}\left( \int_0^T F(v) e^{-\Gamma(v)} d\Gamma(v) \right) + \sum_{i=1}^d \int_0^t \mathbb{1}_{\{\tau_x^i > s-\}} e^{\Gamma(s)} \varphi_i(s) dW_i(s) \\
&\quad - \int_{0+}^t \mathrm{E}^{\mathbb{P}}\left( \int_s^T F(v) e^{\Gamma(s) - \Gamma(v)} d\Gamma(v) \middle| \mathcal{G}_s \right) dM_i^N(s) - \int_0^t \mathbb{1}_{\{\tau_x^i > s\}} F(s) d\Gamma(s),
\end{aligned}$$

where $M_i^N(t) := \mathbb{1}_{\{\tau_x^i \le t\}} - \int_0^t \mathbb{1}_{\{\tau_x^i > s-\}} \mu(s) ds$. On the other hand, we obtain by (A.3) that

$$\mathbb{1}_{\{\tau_x^i \le t\}} F(\tau_x^i) = \int_0^t F(s) d\mathbb{1}_{\{\tau_x^i \le s\}} = \int_0^t F(s) dM_i^N(s) + \int_0^t F(s) \mathbb{1}_{\{\tau_x^i > s\}} d\Gamma(s).$$

Summing up the representations of the two summands from equation (A.4) and using the $\mathcal{F}_T$-measurability of $\mathbb{1}_{\{\tau_x^i \le T\}} F(\tau_x^i)$, we obtain

$$\begin{aligned}
\mathbb{1}_{\{\tau_x^i \le T\}} F(\tau_x^i) &= \mathrm{E}^{\mathbb{P}}\left( \int_0^T F(v) e^{-\Gamma(v)} d\Gamma(v) \right) + \sum_{i=1}^d \int_0^T \mathbb{1}_{\{\tau_x^i > t-\}} e^{\Gamma(t)} \varphi_i(t) dW_i(t) \\
&\quad - \int_{0+}^T \left[ \mathrm{E}^{\mathbb{P}}\left( \int_t^T F(v) e^{\Gamma(t) - \Gamma(v)} d\Gamma(v) \middle| \mathcal{G}_t \right) - F(t) \right] dM_i^N(t).
\end{aligned}$$

Additionally, Corollary 5.1.3 in Bielecki and Rutkowski (2004) also implies that

$$\mathrm{E}^{\mathbb{P}}\left(\int_0^T F(v)e^{-\Gamma(v)}d\Gamma(v)\right) = \mathrm{E}^{\mathbb{P}}\left(\mathbb{1}_{\{\tau_x^i \le T\}}F(\tau_x^i)\right).$$

By the conditional independence assumption on $\tau_x^i,\ i = 1,\ldots,m$, we have in the portfolio case with $\mathbb{F} = \mathbb{G} \vee \bigvee_{i=1}^m \mathbb{I}^i$ that $\mathrm{E}^{\mathbb{P}}\left(\mathbb{1}_{\{\tau_x^i \le T\}}F(\tau_x^i)\middle|\, \mathcal{F}_t\right) = \mathrm{E}^{\mathbb{P}}\left(\mathbb{1}_{\{\tau_x^i \le T\}}F(\tau_x^i)\middle|\, \mathcal{G}_t \vee \mathcal{I}_t^i\right)$. Thus, the statement for the portfolio directly follows by applying the obtained equation to each summand $\mathbb{1}_{\{\tau_x^i \le t\}}F(\tau_x^i),\ i = 1,\ldots,m$, separately and adding the respective decompositions. □

*Proof of Proposition 4.13.* Since $n = d$ and $\det\sigma(t, X(t)) \neq 0$ for all $t \in [0, T^*]$ $\mathbb{P}$-almost surely, the uniqueness of the decompositions follows by Proposition 3.3. Furthermore, Assumption 4.12 implies that $X$ is a Markov process which together with the factorization lemma yields for all cases i) to iv) below that

$$\mathrm{E}^{\mathbb{P}}\left(\cdot|\,\mathcal{G}_t\right) = \mathrm{E}^{\mathbb{P}}\left(\cdot|\,X(t)\right) = f(t, X(t)), \tag{A.5}$$

where $f(t,x) := \mathrm{E}^{\mathbb{P}}\left(\cdot|\,X_t = x\right)$. Define $G(t) := \int_0^t g(s, X(s))ds,\ 0 \le t \le T$, and note that (Shreve, 2004, p. 480)

$$d[G,G](t) = d[G,\Gamma](t) = d[\Gamma,\Gamma](t) = d[G,X_i](t) = d[\Gamma,X_i](t) = 0. \tag{A.6}$$

i) The assumption on the form of $C_0$ together with (A.5) yield that

$$\begin{aligned}\mathrm{E}^{\mathbb{P}}\left(C_0|\,\mathcal{G}_t\right) &= e^{-G(t)}\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T g(s,X(s))ds}h(X(T))\middle|\,\mathcal{G}_t\right) = e^{-G(t)}f(t,X(t))\\ &=: \tilde{f}(t, G(t), X(t)).\end{aligned}$$

Since $f$ is assumed to be smooth, this holds for $\tilde{f}$ as well. Thus, Itô's formula yields for $0 \le t \le T$ (Theorem 33 in Protter, 2005, p. 81)

$$\mathrm{E}^{\mathbb{P}}\left(C_0|\,\mathcal{G}_t\right) - \mathrm{E}^{\mathbb{P}}\left(C_0\right) = \sum_{i=1}^n \int_0^t e^{-G(s)}\frac{\partial f}{\partial x_i}(s, X(s))dM_i^W(s) + \int_0^t a(s)ds,$$

where $a = (a(t))_{0\le t\le T^*}$ is short-hand for all $ds$-quantities. We have used (A.6) and that $(t, G(t), X(t))$ has continuous paths. By the tower property of conditional expectations, the right-hand side $\mathrm{E}^{\mathbb{P}}\left(C_0|\,\mathcal{G}_t\right) - \mathrm{E}^{\mathbb{P}}\left(C_0\right)$ is a martingale. On the other hand, the stochastic integrals with respect to $M_i^W,\ i = 1,\ldots,n$, are martingales as well. Thus, it follows by the uniqueness of the Doob-Meyer decomposition (cf. Theorem 16 in Protter, 2005, p. 116) that the $ds$-term has to vanish. Since $C_0$ is $\mathcal{G}_T$-measurable, the statement follows.

ii) In both cases, $T > t_k$ and $T \le t_k$, we derive the MRT decomposition with the help of Lemma 4.4 and the related remark. Thus, we mainly need to determine the MRT decomposition of $e^{-\Gamma(t_k)}C_{a,k}$ less its expectation.

(a) If $T > t_k$, we consider the decomposition

$$\begin{aligned}&e^{-\Gamma(t_k)}C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\\ &= \left[e^{-\Gamma(t_k)}\mathrm{E}^{\mathbb{P}}\left(C_{a,k}|\,\mathcal{G}_{t_k}\right) - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\right] + e^{-\Gamma(t_k)}\left[C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(C_{a,k}|\,\mathcal{G}_{t_k}\right)\right],\end{aligned} \tag{A.7}$$

and separately derive the MRT decompositions of the two parts.

The assumption on the form of $C_{a,k}$ together with (A.5) yield for $0 \le t \le t_k$ that

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\mathrm{E}^{\mathbb{P}}\left(C_{a,k}\middle|\mathcal{G}_{t_k}\right)\middle|\mathcal{G}_t\right) &= \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\middle|\mathcal{G}_t\right) \\
&= e^{-\Gamma(t)}e^{-G(t)}\mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(t_k)}e^{G(t)-G(T)}h(X(T))\middle|\mathcal{G}_t\right) \\
&= e^{-\Gamma(t)}e^{-G(t)}f^A(t,X(t)) \\
&=: \tilde{f}^A(t,\Gamma(t),G(t),X(t)).
\end{aligned}$$

Since $f^A$ is assumed to be smooth, this holds for $\tilde{f}^A$ as well. Thus, Itô's formula yields for $0 \le t \le t_k$ (Theorem 33 in Protter, 2005, p. 81)

$$\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\mathrm{E}^{\mathbb{P}}\left(C_{a,k}\middle|\mathcal{G}_{t_k}\right)\middle|\mathcal{G}_t\right) - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\mathrm{E}^{\mathbb{P}}\left(C_{a,k}\middle|\mathcal{G}_{t_k}\right)\right) \\
&= \sum_{i=1}^{n}\int_0^t e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))dM_i^W(s) + \int_0^t a(s)ds,
\end{aligned}$$

where $a = (a(t))_{0\le t\le T^*}$ is short-hand for all $ds$-quantities. We have used (A.6) and that $(t,\Gamma(t),G(t),X(t))$ has continuous paths. By the same arguments as in i) the $ds$-term has to vanish, and since $e^{-\Gamma(t_k)}\mathrm{E}^{\mathbb{P}}\left(C_{a,k}\middle|\mathcal{G}_{t_k}\right)$ is $\mathcal{G}_{t_k}$-measurable it follows that

$$e^{-\Gamma(t_k)}\mathrm{E}^{\mathbb{P}}\left(C_{a,k}\middle|\mathcal{G}_{t_k}\right) - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right) = \sum_{i=1}^{n}\int_0^{t_k} e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))dM_i^W(s).$$

Furthermore, applying part i) to $C_{a,k}$ it holds that

$$e^{-\Gamma(t_k)}\left[C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(C_{a,k}\middle|\mathcal{G}_{t_k}\right)\right] = \sum_{i=1}^{n}\int_{t_k}^{T} e^{-\Gamma(t_k)}e^{-G(s)}\frac{\partial f^B}{\partial x_i}(s,X(s))dM_i^W(s).$$

In total, by (A.7) we have

$$\begin{aligned}
&e^{-\Gamma(t_k)}C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right) \\
&= \sum_{i=1}^{n}\int_0^T \Big[e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))\mathbb{1}_{[0,t_k]}(s) \\
&\qquad + e^{-\Gamma(t_k)}e^{-G(s)}\frac{\partial f^B}{\partial x_i}(s,X(s))\mathbb{1}_{(t_k,T]}(s)\Big]dM_i^W(s).
\end{aligned}$$

The statement then follows by Lemma 4.4 and the related remark.

(b) If $T \le t_k$, we consider the decomposition

$$\begin{aligned}
&e^{-\Gamma(t_k)}C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right) \\
&= \left[\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\middle|\mathcal{G}_T\right)C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right)\right] + \left[e^{-\Gamma(t_k)} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\middle|\mathcal{G}_T\right)\right]C_{a,k}
\end{aligned}$$

and again separately derive the MRT decompositions of the two parts. Analogously to above, we obtain

$$\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\middle|\mathcal{G}_T\right)C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right) = \sum_{i=1}^{n}\int_0^T e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))dM_i^W(s),$$

and

$$\left[e^{-\Gamma(t_k)} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}\middle|\mathcal{G}_T\right)\right]C_{a,k} = \sum_{i=1}^{n}\int_T^{t_k} e^{-\Gamma(s)}C_{a,k}\frac{\partial f^B}{\partial x_i}(s,X(s))dM_i^W(s),$$

so that

$$
\begin{aligned}
&e^{-\Gamma(t_k)}C_{a,k} - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(t_k)}C_{a,k}\right) \\
&= \sum_{i=1}^{n}\int_0^{t_k}\Big[e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^A}{\partial x_i}(s,X(s))\mathbb{1}_{[0,T]}(s) \\
&\qquad\qquad + e^{-\Gamma(s)}C_{a,k}\frac{\partial f^B}{\partial x_i}(s,X(s))\mathbb{1}_{(T,t_k]}(s)\Big]dM_i^W(s).
\end{aligned}
$$

The statement then follows by Lemma 4.4 and the related remark.

iii) The assumption on the form of $C_a(v)$ together with (A.5) yield that, for each $v \in [0,T]$,

$$
\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(v)}C_a(v)\middle|\,\mathcal{G}_t\right) &= e^{-\Gamma(t)}e^{-G(t)}\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^v[g(s,X(s))+\mu(s,X(s))]ds}h(X(v))\middle|\,\mathcal{G}_t\right) \\
&= e^{-\Gamma(t)}e^{-G(t)}f^v(t,X(t)) \\
&=: \tilde{f}^v(t,\Gamma(t),G(t),X(t)),\ \ t \le v,
\end{aligned}
$$

where $f^v : [0,v]\times\mathbb{R}^n \to \mathbb{R}$. Since $f^v$ is assumed to be smooth, this holds for $\tilde{f}^v$ as well. Thus, Itô's formula yields for $t \le v$ (Protter, 2005, p. 81)

$$
\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(v)}C_a(v)\middle|\,\mathcal{G}_t\right) - \mathrm{E}^{\mathbb{P}}\left(e^{-\Gamma(v)}C_a(v)\right) \\
&= \sum_{i=1}^{n}\int_0^t e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f^v}{\partial x_i}(s,X(s))dM_i^W(s) + \int_0^t a(s)ds,
\end{aligned}
$$

where $a = (a(t))_{0\le t\le T^*}$ is short-hand for all $ds$-quantities. We have used (A.6) and that $(t,\Gamma(t),G(t),X(t))$ has continuous paths. By the same arguments as in i), the $ds$-term has to vanish. Thus, we obtain by Lemma 4.6 and the related remark that

$$
\begin{aligned}
\psi_i^W(t) &= (m-N(t-))e^{\Gamma(t)}\int_t^T \varphi_i^v(t)dv \\
&= (m-N(t-))e^{\Gamma(t)}\int_t^T e^{-\Gamma(t)}e^{-G(t)}\frac{\partial f^v}{\partial x_i}(t,X(t))dv \\
&= (m-N(t-))e^{-G(t)}\int_t^T \frac{\partial f^v}{\partial x_i}(t,X(t))dv,\ \ i=1,\ldots,n,
\end{aligned}
$$

and

$$
\begin{aligned}
\psi^N(t) &= -\int_t^T \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(v)}C_a(v)\middle|\,\mathcal{G}_t\right)dv \\
&= -\int_t^T e^{-G(t)}\mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^v[g(s,X(s))+\mu(s,X(s))]ds}h(X(v))\middle|\,\mathcal{G}_t\right)dv \\
&= -e^{-\int_0^t g(s,X(s))ds}\int_t^T f^v(t,X(t))dv.
\end{aligned}
$$

iv) Since all involved processes are $\mathbb{G}$-adapted, and by the assumption on the form of $C_{ad}(t)$ and $\mu(t)$

together with (A.5), it follows that for $0 \leq t \leq T$

$$
\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left(\int_0^T C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds \,\middle|\, \mathcal{G}_t\right)\\
&= \int_0^t C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds + \mathrm{E}^{\mathbb{P}}\left(\int_t^T C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds \,\middle|\, \mathcal{G}_t\right)\\
&= \int_0^t C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds + e^{-\Gamma(t)}e^{-G(t)}\mathrm{E}^{\mathbb{P}}\left(\int_t^T e^{\Gamma(t)-\Gamma(s)}e^{G(t)-G(s)}h(s,X(s))\mu(s)ds \,\middle|\, \mathcal{G}_t\right)\\
&= I(t) + e^{-\Gamma(t)}e^{-G(t)}f(t,X(t))\\
&=: \tilde{f}(t,\Gamma(t),G(t),X(t),I(t)),
\end{aligned}
$$

where $I(t) := \int_0^t C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds$. Since $f$ is assumed to be smooth, this holds for $\tilde{f}$ as well. Thus, Itô's formula yields for $0 \leq t \leq T$ (cf. Theorem 33 in Protter, 2005, p. 81)

$$
\begin{aligned}
&\mathrm{E}^{\mathbb{P}}\left(\int_0^T C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds \,\middle|\, \mathcal{G}_t\right) - \mathrm{E}^{\mathbb{P}}\left(\int_0^T C_{ad}(s)e^{-\Gamma(s)}\mu(s)ds\right)\\
&= \sum_{i=1}^n \int_0^t e^{-\Gamma(s)}e^{-G(s)}\frac{\partial f}{\partial x_i}(s,X(s))dM_i^W(s) + \int_0^t a(s)ds,
\end{aligned}
$$

where $a = (a(t))_{0\leq t\leq T^*}$ is short-hand for all $ds$-quantities. We have used (A.6),

$$
d[I,\Gamma](t) = d[I,G](t) = d[I,X_i](t) = 0
$$

(Shreve, 2004, p. 480), and that $(t,\Gamma(t),G(t),X(t))$ has continuous paths. By the same arguments as in i) the $ds$-term has to vanish. Since analogously to Lemma 4.4 it can be shown that Lemma 4.8 also holds with respect to $dM_i^W$-integrals instead of $dW_i$-integrals, the statement follows.

□

*Proof of Proposition 4.17.* Note that any conditional expectation $\mathrm{E}^{\mathbb{P}}(\cdot|\mathcal{G}_t)$ is predictable, since it is by definition $\mathcal{G}_t$-measurable and $\mathcal{G}_t$ is left-continuous as a result of the continuity of Brownian motions.

i) Define $\psi_{ak}^N(t) := \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\, \mathcal{G}_t\right)$ for all $t \in [0,t_k]$. According to the introductory comment, the process $(\psi_{ak}^N(t))_{0\leq t\leq t_k}$ is predictable. Furthermore, applying Jensen's inequality for conditional expectations (cf. Protter, 2005, p. 11), and using that $\Gamma(t)$ is non-decreasing in $t$, it follows that

$$
\begin{aligned}
&\sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\left[\psi_{ak}^N(t)\right]^4\right) = \sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\left[\mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\middle|\, \mathcal{G}_t\right)\right]^4\right)\\
&\leq \sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\mathrm{E}^{\mathbb{P}}\left(\left[e^{\Gamma(t)-\Gamma(t_k)}C_{a,k}\right]^4\middle|\, \mathcal{G}_t\right)\right) \leq \sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\mathrm{E}^{\mathbb{P}}\left(\left[C_{a,k}\right]^4\middle|\, \mathcal{G}_t\right)\right)\\
&= \sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\left[C_{a,k}\right]^4\right) = \mathrm{E}^{\mathbb{P}}\left(\left[C_{a,k}\right]^4\right) < \infty \quad \text{(by assumption)}.
\end{aligned}
$$

Since we also assume that $\sup_{t\in[0,t_k]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$, the statement follows by Lemma 4.16.

ii) Define $\psi_a^N(t) := \int_t^T \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)ds$ for all $t \in [0,T]$. According to the introductory comment, the process $(\psi_a^N(t))_{0\leq t\leq T}$ is predictable. Furthermore, since $0 \leq e^{\Gamma(t)-\Gamma(s)} \leq 1$ for $s \geq t$ and since $C := \sup_{0\leq t\leq T} \mathrm{E}^{\mathbb{P}}\left(|C_a(t)|\right) < \infty$ as a result of the boundedness of $C_a(t)$, it follows by applying Jensen's inequality for integrals and for conditional expectations (for the latter, cf. Protter, 2005, p. 11) that for any $t \in [0,T]$

$$\left|\psi_a^N(t)\right| = \left|\int_t^T \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)ds\right| \leq \int_t^T \left|\mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(s)}C_a(s)\middle|\mathcal{G}_t\right)\right|ds$$
$$\leq \int_t^T \mathrm{E}^{\mathbb{P}}\left(e^{\Gamma(t)-\Gamma(s)}\left|C_a(s)\right|\middle|\mathcal{G}_t\right)ds \leq C\,T.$$

Thus, we have

$$\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left[\psi_a^N(t)\right]^4\right) \leq \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left([CT]^4\right) = C^4\,T^4 < \infty.$$

Since we also assume that $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$, the statement follows by Lemma 4.16.

iii) Since $X_m, Y_m, X, Y \in L^2(\mathbb{P})$ and $X_m \xrightarrow{L^2} X,\ Y_m \xrightarrow{L^2} Y$ implies that $X_m + Y_m \xrightarrow{L^2} X + Y$, it is sufficient to show that

a) $\displaystyle \frac{1}{m}\int_{0+}^T \left[-\mathrm{E}^{\mathbb{P}}\left(\int_t^T C_{ad}(s)e^{\Gamma(t)-\Gamma(s)}\mu(s)ds\middle|\mathcal{G}_t\right)\right]dM^N(t) \xrightarrow[m\to\infty]{L^2} 0,$ and

b) $\displaystyle \frac{1}{m}\int_{0+}^T C_{ad}(t)dM^N(t) \xrightarrow[m\to\infty]{L^2} 0.$

Define $\psi_{ad,1}^N(t) := -\mathrm{E}^{\mathbb{P}}\left(\int_t^T C_{ad}(s)e^{\Gamma(t)-\Gamma(s)}\mu(s)ds\middle|\mathcal{G}_t\right)$ and $\psi_{ad,2}^N(t) := C_{ad}(t)$ for all $t \in [0,T]$. Note that since by assumption $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^4(t)\right) < \infty$, it also follows by Jensen's inequality that

$$\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) \leq \sup_{t\in[0,T]} \sqrt{\mathrm{E}^{\mathbb{P}}\left(\mu^4(t)\right)} < \infty.$$

ad a): According to the introductory comment, the process $(\psi_{ad,1}^N(t))_{0\leq t\leq T}$ is predictable. Since $0 \leq e^{\Gamma(t)-\Gamma(s)} \leq 1$ for $s \geq t$ and since $C_1 := \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left|C_{ad}(t)\right|\right) < \infty$ as a result of the boundedness of $C_{ad}(t)$, it follows by applying Jensen's inequality for integrals and for conditional expectations (for the latter, cf. Protter, 2005, p. 11) that

$$\left|\psi_{ad,1}^N(t)\right| = \left|\mathrm{E}^{\mathbb{P}}\left(\int_t^T C_{ad}(s)e^{\Gamma(t)-\Gamma(s)}\mu(s)ds\middle|\mathcal{G}_t\right)\right|$$
$$\leq \mathrm{E}^{\mathbb{P}}\left(\int_t^T \left|C_{ad}(s)\right|e^{\Gamma(t)-\Gamma(s)}\mu(s)ds\middle|\mathcal{G}_t\right)$$
$$\leq C_1\,\mathrm{E}^{\mathbb{P}}\left(\int_t^T \mu(s)ds\middle|\mathcal{G}_t\right) \leq C_1\,\mathrm{E}^{\mathbb{P}}\left(\int_0^T \mu(s)ds\middle|\mathcal{G}_t\right).$$

Since by assumption $C_2 := \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^4(t)\right) < \infty$, this implies

$$\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left[\psi_{ad,1}^N(t)\right]^4\right) \leq \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left[C_1\,\mathrm{E}^{\mathbb{P}}\left(\int_0^T \mu(s)ds\middle|\mathcal{G}_t\right)\right]^4\right)$$
$$\stackrel{(*)}{\leq} \sup_{t\in[0,T]} C_1^4\,\mathrm{E}^{\mathbb{P}}\left(\mathrm{E}^{\mathbb{P}}\left(\int_0^T \mu^4(s)ds\middle|\mathcal{G}_t\right)\right) = \sup_{t\in[0,T]} C_1^4\,\mathrm{E}^{\mathbb{P}}\left(\int_0^T \mu^4(s)ds\right)$$
$$\stackrel{(**)}{=} \sup_{t\in[0,T]} C_1^4 \int_0^T \mathrm{E}^{\mathbb{P}}\left(\mu^4(s)\right)ds \leq C_1^4\,C_2\,T < \infty.$$

where $(*)$ again follows by Jensen's inequality for integrals and conditional expectations and $(**)$ from the theorem of Fubini-Tonelli. Since $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$ as shown above, the statement follows by Lemma 4.16.

ad b): By assumption the process $(\psi^N_{ad,2}(t))_{0\leq t\leq T}$ is predictable. As a result of the boundedness of $C_{ad}(t)$, it also holds $C_1 := \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(|C_{ad}(t)|\right) < \infty$, so that

$$\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\left[\psi^N_{ad,2}(t)\right]^4\right) = \sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left([C_{ad}(t)]^4\right) \leq C_1^4 < \infty.$$

Since $\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left(\mu^2(t)\right) < \infty$ as shown above, the statement follows by Lemma 4.16.

□

*Proof of Proposition 4.19.* Since $M_i^W(t) = \sum_{k=1}^d \int_0^t \sigma_{ik}(s)dW_k(s),\ 0 \leq t \leq T^*$, it follows that

$$R^m_{i,\cdot} = \sum_{k=1}^d \sum_{j=1}^d \int_0^T \left[(m - N(t-))e^{\Gamma(t)}\mathbb{1}_{[0,t_k]}(t) + (m - N(t_k))e^{\Gamma(t_k)}\mathbb{1}_{(t_k,T]}(t)\right] \times \varphi_{j,\cdot}(t)\sigma^{-1}_{ji}(t)\sigma_{ik}(t)dW_k(t). \tag{A.8}$$

Because of the additivity of integration and the continuous mapping theorem, it is sufficient to prove the convergence of each summand in (A.8), $i = 1,\ldots,n,\ j,k = 1,\ldots,d$, separately. For this, by Lemma 4.18, we only need to show that each $\varphi_{j,\cdot}(t)\sigma^{-1}_{ji}(t)\sigma_{ik}(t)$ is $\mathbb{G}$-adapted and continuous. We have:

- By assumption, $\sigma(t)$ is $\mathbb{G}$-adapted with continuous paths.
- When determining the inverse of $\sigma(t)$ with Cramer's rule and the necessary determinants with Laplace's formula, it can be seen that $\sigma^{-1}_{ij}(t)$ is a continuous function of the matrix components $\sigma_{ij}(t),\ i = 1,\ldots n, j = 1\ldots,d$. So $\sigma^{-1}_{ij}(t)$ has itself continuous paths and is $\mathbb{G}$-adapted.
- In all parts ii), iii) and iv), $\varphi_{j,\cdot}(t)$ is a conditional expectation of the form $\mathrm{E}^{\mathbb{P}}\left(\cdot|\,\mathcal{G}_t\right)$ or can be transformed into such an expectation. As a result, $\varphi_{j,\cdot}(t)$ is by definition $\mathbb{G}$-adapted. Furthermore, it follows by the martingale representation theorem that it is also continuous.

As a result, the product $\varphi_{j,\cdot}(t)\sigma^{-1}_{ji}(t)\sigma_{ik}(t)$ is also $\mathbb{G}$-adapted and continuous, and the statement follows by Lemma 4.18. □