Crisis and contagion in financial networks: a dynamic approach

Dimitri Bouche, Hubert Kempf and Arnaud Lionnet

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Abstract

We study the dynamics of a connected banking sector where the financial links between banks are explicitly modelled, including the liquidation procedures in the case of the failure of an individual bank. This model of banking network allows us to simulate the bankruptcy contagion process and quantify the extent of a banking crisis. We discuss the impact of various structural parameters on the contagion process.

Contents

1 Introduction

Systemic risk is a poignant real world issue — and its complexity calls for numerical simulations. The recent advances in network theory and its applications to economics (see Jackson 2008, Gai and Papadai 2010, Acemoglu and Ozdagla 2015, Elliott, Golub and Jackson 2014) make it a potentially useful tool to study this phenomonon. Such network-based models are by definition complex and as a consequence we can rarely hope to resolve them analytically. They however enable us to understand what the macro implication of entangled micro-phenomenon are. As a consequence despite their complexity, they are of great interest to the field of economics.

Approaching systemic risk in such manner is particularly tempting since the collapse of Lehman Brothers has shown in 2008 that the global financial system was fragile and prone to contagion. The intertwined balance sheets and complex derivatives commitments between banks are among the possible culprits, for they seem to have lead the global financial system to act as an amplifier of economics issues. One default turned out to spread like a disease shaking the whole financial system to its core and bringing about a fierce economic crisis. How did this one default turned into a systemic event? Systemic risk can be seen as network issue since as stated above, banks are interdependant on one another in a lots of ways. However, in order to simplify, we may model only one type of link between them—loans and debts. This generates a bank network which nodes are the banks while weighted edges represent the loans/debts between them. This was done in Yao, LIu and Zhiang (2016) where the authors conclude that some networks are more prone to network-wide contagion—systemic events—than other.

Our aim is to create a more advanced model of financial networks that can be used for any structure of network. Each node's state is given by a simplified balance sheet from which an equity can be computed. This is the key quantity that determines whether a node is in default. Since we want to only consider solvency defaults and not liquidity default, we design a model such that the latter cannot happen. We then simulate and study the propagation of defaults brought about by random returns from risky assets included in the balance sheets. However we do so in a dynamic fashion. In that sense, we go one step further in comparison to the existing literature where dynamics is often limited either to a fixed small number of periods (as in Acemoglu and Ozdagla 2015) or to a "killing cascade" (Gai and Papadai 2010, Elliott, Golub and Jackson 2014) — after an initial shock banks may default which may in turn bring about new defaults and so on until no more banks are defaulting. Having a real time dynamics will give the possibility to study several sequential control policies such as dynamic resource allocations (along the line of Argyris, Scaman and Vayatis 2016) . The final long term objective being to find a policy that is likely to contain potential systemic events—events involving a significant part of the system—under constraints of budget.

2 Economic description of the model

We focus our analysis on banks and the only type of credit we consider is inter-bank loans. Banks do not take stakes in one another. The outside economy is simply modelled as a set of risky assets which yields random returns, they could represent for instance loans to companies, to individuals, investment in public projects... as well as financial products.

2.1 Banks

2.1.1 Balance sheet representation

Let n be the number of banks in the network. At each time t, bank i is represented as a simplified balance sheet :

Assets ($\mathcal{A}_t^i$)	Liabilities ($\mathcal{L}_t^i$)
Reserve (R_t^i) Inter-bank loans (L_t^i) Portfolio (P_t^i)	Equity (E_t^i) Inter-bank debts (D_t^i)

- On the assets side :
  - Reserve R_t^i is the amount of reserves, it is perfectly liquid (equivalent to cash).
  - L_t^i corresponds to the cumulative face-value of the loans bank i granted other banks. Also, let us denote by L_t^{ij} the face-value of the loan bank i granted bank j. Thus $L_t^i = \sum_{j=1}^n L_t^{ij}$.
  - Portfolio P_t^j is the amount of money invested in the economy.
- On the liabilities side :
  - Equity E_t^i is equal to the assets minus the liabilities.
  - D_t^i corresponds to the cumulative face-value of the loans other banks granted bank i. Thus $D_t^i = \sum_{k=1}^n L_t^{ki}$.

Let us denote by :

- $L_t \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ the matrix of inter-bank loans which entry (i,j) is L_t^{ij}. At this point, we should highlight that:
  - In order to simplify the graph as much as possible, loans and debts will be netted, which means that $L_t^{ij} > 0 \Rightarrow L_t^{ji} = 0$. Given a matrix L_t, we can obtain its netted version L'_t using component-wise maximum by doing the following operation :

    $$L'_t = \max\left(\mathbf{0}_{n,n}, L_t - L_t^T\right).$$

    Where $\mathbf{0}_{n,n} \in \mathcal{M}_{n,n}(\mathbb{R})$ is the matrix full of zeros.

- The matrix L_t contains all the information on the graph at time t, thus no need to have a debt matrix since it is simply equal to L_t^T, the transposed version of L_t.

- $E_t \in \mathcal{M}_{n,1}(\mathbb{R})$ the vector of equities which i-th element is E_t^i.
- $R_t \in \mathcal{M}_{n,1}(\mathbb{R})$ the vector of reserves which i-th element is R_t^i.

2.1.2 Interest rates

We introduce the inter-bank interest rate r_t^i which determines the cost of borrowing for bank i at time t. Its existence is justified by the extra risk taken when lending money. We make several assumptions for now which we may relax in a more complex version:

- Interest rates are deterministic.
- r_t^j does not depend on time, thus we will omit the time-index $r_t^i = r^i$.
- $\forall i \in [\![1, n]\!], \quad r^i = r$, thus every bank can borrow money for the same interest rate r to other banks.
- Moreover, $1 > r > 0$.

2.2 Portfolios and risky assets

2.2.1 Risky assets

There are m risky assets in the economy. We take a wide definition of risky assets which entails productive investments such as loans to companies or individuals, investments in public projects... Thus risky assets are not necessarily stocks or financial products although they can be. The only fundamental conditions to match the definition are:

- to be outside of the network of banks.
- to be risky to some extent.

Each risky asset has a time-dependent valuation. For a given $l \in [\![1, m]\!]$, let us denote by X_t^l this valuation at date t with time being discrete.

For now, those risky assets do not yield dividend. As a consequence gains (resp. losses) only come from increases (resp. drops) in the valuations.

We model the movements in valuations using Gaussian increases:

$$X_t^l = X_{t-1}^l + \omega_{t-1}^l.$$

With:

$$\omega_{t-1}^l \sim \mathcal{N}\left(\mu_l, \sigma_l^2\right).$$

We denote by ω_{t-1} the vector of increases in t. Since its components are Gaussian and independent, this is a Gaussian vector with:

- Mean vector $\mu \in \mathcal{M}_{m,1}(R)$
- Covariance matrix $\Sigma = Diag\left[\left(\sigma_l^2\right)_{1\leq l\leq m}\right] \in \mathcal{M}_{m,m}\left(\mathbb{R}^+\right)$.

In mathematical writing:

$$\omega_{t-1} \sim \mathcal{N}(\mu, \Sigma).$$

To finish with, we make the assumption that increases are independent across time:

$$\forall t \neq t',\ w_{t-1} \perp\!\!\!\perp w_{t-1'}.$$

2.2.2 Banks' portfolios and valuation

Banks invest in those risky assets. Let us denote by $Q_t^i \in \mathcal{M}_{1,m}\left(\mathbb{R}^+\right)$ the vector which entry Q_t^{il} is number of unit of product l that bank i holds in its portfolio at time t. The matrix $Q_t \in \mathcal{M}_{n,m}\left(\mathbb{R}^+\right)$ is the matrix which i-th row is Q_t^i.

As a consequence, the value of i's portfolio is simply given by the dot product: $P_t^i = Q_t^i X_t$. We can aggregate this formula for all banks in matrix form:

$$P_t = Q_t X_t.$$

With $P_t \in \mathcal{M}_{n,1}\left(\mathbb{R}^+\right)$ being the portfolio vector.

3 General dynamics of the system

3.1 Sequence of events

In this subsection, we establish the chronology of events. Since several operations take place within a given time-lapse t, the ordering of events need to be precised. We will detail the equations later—in what follows, we make reference at each stage description to the appropriate section/subsection.

- **Stage 1: updates** (detailed in 3.3). At the beginning of this stage, the state of a bank is given by the vector $\left(E_{t-1}^i, D_{t-1}^i, R_{t-1}^i, P_{t-1}^i, L_{t-1}^i\right)$. The following operations are then carried out:
  - If banks have defaulted in the previous period, the defaults are now effective and the creditors of the defaulted banks suffer the corresponding losses.
  - The reserves are updated: banks pay interest rates on inter-bank loans and if banks have defaulted in the previous period, proceedings from liquidation are added to the reserves.

  - The valuation of the risky assets are updated and the value of portfolios are changed accordingly.

  At the end of this stage, the state of a bank is given by a vector of five variables : $\left(\widehat{E}_t^i, \widehat{D}_t^i, \widehat{R}_t^i, \widehat{P}_t^i, \widehat{L}_t^i\right)$.

- **Stage 2: checking for default**. Banks for which $E_t^i \leq \bar{E}^i$ declare default and are liquidated (see section 4 for the detailed processes of liquidation). Although their defaulting is not public information yet, it will become so at the beginning of the next period. The vector $\bar{E}$ which components are the $\bar{E}^i$ is a minimal threshold value for the equity of each bank. It enables us to implicitly include deposits from investors or individuals from outside the system in the balance sheets of the banks.

- Stage 2: **defaulting stage** (detailed in ...).

  In this stage the fulfillment of the default rule (to be set and discussed later) is checked for all banks in the network.

- **Stage 3: balance sheet management** (detailed in 3.4). Banks readjust between portfolio and reserves according to (detailed in 3.2):

  - The reserve rule
  - The financial choice for the portfolio.

  At the end of the stage, the state of a bank is given by the 5 state variables $\left(E_t^i, D_t^i, R_t^i, P_t^i, L_t^i\right)$.

3.2 Viability conditions

We need to introduce additional hypothesis, initial conditions and constraints in order to maintain our model's coherence.

- We assume that banks cannot have a negative expected variation of equity. Even though they are unidentified and not included in our model, we can assume that there are shareholders who own the banks. They indeed want their shares to gain value which justify our hypothesis on the variation of equity. This is the subject of section 3.2.1.

- If some banks have borrowed more than they have lent so as to invest in their portfolio section, their reserves will deplete mechanically after a given number of periods. As a consequence, we need to define transfer rules between portfolio and reserve. We thus introduce the reserve rule and the financial choice in 3.2.2.

- We make the assumption that banks can sell assets from their portfolio freely as a regular operation of management without paying any fees or having any price impact.

This seems to be a reasonable hypothesis since only small amounts of risky assets are bought and sold in a usual financial market context in order to obtain the chosen portfolio. To finish with it is worth mentioning that we do not impose integer-valued quantities.

3.2.1 Bank should have a positive expected net-worth delta

Expected returns vs interest rates. Since risky assets' returns are uncertain, investment opportunities must offer a risk premium—although loans to other banks are risky too they are obviously less so. This implies a first initial condition on the means μ of the vector of increases in relation to the initial prices:

$$\forall l \in [\![1, m]\!],\ \frac{\mu_l}{X_0^l} \geq r.$$

Balance sheets coherence. As a consequence, it can be interesting for a bank to borrow money from other banks in order to invest. Although it may do so only in such a way that it gains money on average. Denoting by $\mathcal{F}_t$ the information available up to time t, this amounts to:

$$\mathbb{E}\left[E_t^i | \mathcal{F}_{t-1}\right] \geq E_{t-1}^i.$$

Let us remark first that since $E_{t-1}^i \geq \bar{E}^i$, this implies that a bank that has not defaulted in $t-1$ cannot be expected to default in t.

Applying expectation conditionally on $\mathcal{F}_t$ to (6) we get:

$$\mathbb{E}\left[E_t^i | \mathcal{F}_{t-1}\right] = E_{t-1}^i + rL_{t-1}^i - rD_{t-1}^i + Q_{t-1}^i \mathbb{E}\left[\omega_t^l | \mathcal{F}_{t-1}\right].$$

Which is equivalent to:

$$\mathbb{E}\left[E_t^i | \mathcal{F}_{t-1}\right] - E_{t-1}^i = rL_{t-1}^i - rD_{t-1}^i + Q_{t-1}^i \mu.$$

As a consequence,

$$\mathbb{E}\left[E_t^i | \mathcal{F}_{t-1}\right] \geq E_{t-1}^i \Leftrightarrow rL_{t-1}^i - rD_{t-1}^i + Q_{t-1}^i \mu \geq 0.$$

Rearranging the terms, the condition is:

$$Q_{t-1}^i \mu \geq r\left(D_{t-1}^i - L_{t-1}^i\right).$$

Actually this condition may prove too difficult to enforce at each future period. We will thus consider only its initial version:

$$Q_0^i \mu \geq r\left(D_0^i - L_0^i\right). \tag{1}$$

3.2.2 Regulation rules and management conditions

Reserve rule We want to ensure that banks can pay their interest rates the next day. We thus introduce the following regulatory rule imposed by some prudential authority. We take the positive part of the second term since lending more than one has borrowed does not grant one the right to have negative reserves.

$$R_t^i + r \max\left(0, \left(L_t^i - D_t^i\right)\right) \geq 0. \tag{2}$$

Proposition A bank that has not defaulted in t necessarily has enough liquidity in t to comply with the reserve rule.

To prove this, let us distinguish between two cases:

- $\widehat{D}_t^i \leq \widehat{L}_t^i$. In that case, a bank net interest rate cash flows are positive and as a consequence it automatically complies with the reserve rule since its reserves are positive (we do not authorize negative reserves). In other words, since $R_t^i \geq 0$

$$\widehat{L}_t^i - \widehat{D}_t^i \geq 0 \Rightarrow r\left(\widehat{L}_t^i - \widehat{D}_t^i\right) \geq 0 \Rightarrow \widehat{R}_t^i + r\left(\widehat{L}_t^i - \widehat{D}_t^i\right)) \geq 0.$$

- $\widehat{D}_t^i > \widehat{L}_t^i$. The bank has not defaulted in t, thus:

$$\widehat{E}_t^i > \bar{E}^i \Rightarrow \widehat{E}_t^i > 0 \Leftrightarrow \widehat{R}_t^i + \widehat{P}_t^i + \widehat{L}_t^i - \widehat{D}_t^i > 0.$$

  We have:

$$\widehat{D}_t^i - \widehat{L}_t^i > \widehat{R}_t^i + \widehat{P}_t^i.$$

  Since $r < 1$:

$$\widehat{D}_t^i - \widehat{L}_t^i > r\left(\widehat{D}_t^i - L_t^i\right).$$

  As a consequence:

$$\widehat{R}_t^i + \widehat{P}_t^i > r\left(\widehat{D}_t^i - \widehat{L}_t^i\right).$$

We can thus conclude that a bank that has not defaulted in t always has enough liquidity to be able to reallocate its liquid assets in t so as to comply with the reserve rule. It can do so by selling partially its portfolio in order to increase its reserves.

This rule enables us to avoid liquidity defaults, and being solvent is a sufficient condition to be able to comply with it. Since we want to account only for solvency defaults, our model is coherent. Indeed, default can be defined as either having not enough cash to honor one's financial obligation (liquidity default) or not having enough equity (solvency default)—or both. We have shown here that since the only illiquid assets are inter-bank loans, if equity is positive in t then by construction liquidity default cannot happen in t.

Portfolio management condition Since the value of the assets in the portfolio will vary, so will the size of the portfolio in relation to the rest of the balance sheet quantities. As a basic rule of management, we may state that each bank wants to maximize its investment in the portfolio under the constraint that it amounts to a given fraction of the total of its assets $\mathcal{A}_t$ (and that it is in accordance with the reserve rule). Let us define then by α^i the target investment percentage of bank i which we can interpret as a behavioral parameter. We assume for now that the share of wealth invested in each asset remains constant. As a consequence, we can formulate the problem in term of P_t^i only :

$$\max P_t^i$$

s.t.

$$P_t^i \leq \alpha^i \mathcal{A}_t^i$$

$$R_t^i + \max\left(0,\ r\left(L_t^i - D_t^i\right)\right) \geq 0$$

$$P_t^i + R_t^i = \widehat{P}_t^i + \widehat{R}_t^i$$

3.3 Stage 1: updates

3.3.1 Reserves

The evolution of reserves is given by:

$$\widehat{R}_t^i = R_{t-1}^i + r\widehat{L}_t^i - r\widehat{D}_t^i. \tag{3}$$

3.3.2 Portfolio

By definition:

$$\widehat{P}_t^i = Q_{t-1}^i X_t.$$

Using the dynamics of the risky assets this is equivalent to:

$$\widehat{P}_t^i = Q_{t-1}^i \left(X_{t-1} + \omega_{t-1}\right).$$

Supposing that bank i has not defaulted in $t-1$:

$$\widehat{P}_t^i = P_{t-1}^i + Q_{t-1}^i \omega_t. \tag{4}$$

3.3.3 Equity

Mathematically, the accounting definition of equity is:

$$\widehat{E}_t^i = \widehat{R}_t^i + \widehat{P}_t^i + \widehat{L}_t^i - \widehat{D}_t^i. \tag{5}$$

Putting all above dynamic equation together yields:

$$\widehat{E}_t^i = R_{t-1}^i + r\widehat{L}_t^i - r\widehat{D}_t^i + \widehat{L}_t^i - \widehat{D}_t^i + P_{t-1}^i + Q_{t-1}^i\omega_t.$$

If no bank has defaulted in $t-1$, $\widehat{L}_t^i = L_{t-1}^i$ and $\widehat{D}_t^i = D_{t-1}^i$ which implies:

$$E_t^i = R_{t-1}^i + P_{t-1}^i + L_{t-1}^i - D_{t-1}^i + r\left(L_{t-1}^i - D_{t-1}^i\right) + Q_{t-1}^i\omega_t.$$

We can thus deduce the following recursion formula for equity if no bank has defaulted in the previous period:

$$\Leftrightarrow E_t^i = E_{t-1}^i + r\left(\widehat{L}_{t-1}^i - \widehat{D}_{t-1}^i\right) + Q_{t-1}^i\omega_t. \tag{6}$$

3.4 Stage 3 : balance sheet management

3.4.1 Reserve rule and portfolio rule in practice

Let us distinguish two cases:

- $\widehat{R}_t^i < \max\left(0,\ r\left(\widehat{D}_t^i - \widehat{L}_t^i\right)\right)$ (reserve rule not matched).
- $\widehat{R}_t^i \geq \max\left(0,\ r\left(\widehat{D}_t^i - \widehat{L}_t^i\right)\right)$ (reserve rule matched).

Let us also define the portfolio valuation objective P_t^i that the bank wishes to reach.

Reserve rule matched No need to rebalance to comply with the reserve rule. We can increase the portfolio valuation objective although when doing so, we must keep in mind the reserve rule. We distinguish again different cases:

- $\widehat{P}_t^i > \alpha^i\mathcal{A}_t^i$. In that case, a portion $\widehat{P}_t^i - \alpha^i\mathcal{A}_t^i$ of the portfolio must be sold. No other actions are required.

$$P_t^i = \alpha^i\mathcal{A}_t^i.$$

- $\widehat{P}_t^i \leq \alpha^i\mathcal{A}_t^i$. In that case the bank wants to expand its portfolio so as to saturate the constraint if possible given the reserve rule.

$$P_t^i = \min\left(\alpha^i\mathcal{A}_t^i,\ \ \widehat{P}_t^i + \widehat{R}_t^i - \max\left(0,\ r\left(\widehat{D}_t^i - \widehat{L}_t^i\right)\right)\right).$$

Reserve rule not matched The bank have to sell a portion of its portfolio to comply with the reserve rule. However, it may sell more of it if it is still too large according to the financial choice. Let us distinguish two cases:

- $\widehat{P}_t^i+\widehat{R}_t^i-\max\left(0,\ r\left(\widehat{D}_t^i-\widehat{L}_t^i\right)\right)\leq \alpha^i\mathcal{A}_t^i$. In that case, the bank sells only the amount necessary to comply with the reserve rule — since $\widehat{R}_t^i-\max\left(0,\ r\left(\widehat{D}_t^i-\widehat{L}_t^i\right)\right)<0$ this is indeed selling . As a consequence:

$$P_t^i=\widehat{P}_t^i+\widehat{R}_t^i-\max\left(0,\ r\left(\widehat{D}_t^i-\widehat{L}_t^i\right)\right).$$

- $\widehat{P}_t^i+\widehat{R}_t^i-\max\left(0,\ r\left(\widehat{D}_t^i-\widehat{L}_t^i\right)\right)>\alpha^i\mathcal{A}_t^i$. In that case, the bank sells enough portfolio assets to comply with the financial choice. Which is enough to comply also with the reserve rule since we have: $\widehat{P}_t^i-\alpha^i\mathcal{A}_t^i>-\widehat{R}_t^i+\max\left(0,\ r\left(\widehat{D}_t^i-\widehat{L}_t^i\right)\right)$. As a consequence:

$$P_t^i=\alpha^i\mathcal{A}_t^i.$$

Synthesis We can actually unify all four cases in the following formula:

$$P_t^i=\min\left(\widehat{P}_t^i+\widehat{R}_t^i-\max\left(0,\ r\left(\widehat{D}_t^i-\widehat{L}_t^i\right)\right),\ \alpha^i\mathcal{A}_t^i\right). \tag{7}$$

The new amount of reserves is also deduced easily from P_t^i:

$$R_t^i=\widehat{R}_t^i+\widehat{P}_t^i-P_t^i. \tag{8}$$

3.4.2 Find quantities to match a given portfolio valuation objective

Now that P_t^i is known, we show here how to adjust the quantities invested in each asset so as to match this portfolio valuation objective. Given prices X_t and quantities Q_{t-1}^i, we seek to find the vector of quantities Q_t^i for which $Q_t^iX_{t-1}=P_t^i$ while keeping the relative quantities constant.

Given a portfolio value objective P_t^i, a portfolio current valuation $\widehat{P}_t^i$, prices X_{t-1} and quantities Q_{t-1}^i, we seek to find the vector of quantities Q_t^i for which $Q_t^iX_{t-1}=P_t^i$ while keeping the relative quantities constant i.e:

$$\forall l\in[\![1,m]\!],\quad \frac{Q_t^{il}}{\sum_{c=1}^m Q_t^{ic}}=\frac{Q_{t-1}^{il}}{\sum_{c=1}^m Q_{t-1}^{ic}}$$

Let us define:

$$Q_{t-1}^{\bar{i}}=\sum_{c=1}^m Q_{t-1}^{ic}$$

$$\delta^{il} = \frac{Q^{il}_{t-1}}{Q^{\bar{i}}_{t-1}}$$

It is easy to verify that given the constraints, we only have one degree of freedom to modify the quantities and thus finding Q^i_t boils down to finding η^i such that:

$$\sum_{l=1}^{m} \left(Q^{il}_{t-1} + \delta^{il}\eta^i \right) X^l_t = P^i_t$$

The resulting solution quantities being:

$$\forall l, \quad Q^{il}_t = \left(Q^{il}_{t-1} + \delta^{il}\eta^i \right)$$

Developing and solving in η^i yields:

$$\eta^i = Q^{\bar{i}}_{t-1} \frac{P^i_t - \widehat{P}^i_t}{\widehat{P}^i_t}$$

Thus:

$$\forall l, \quad Q^{il}_t = Q^{il}_{t-1} \left(1 + \frac{P^i_t - \widehat{P}^i_t}{\widehat{P}^i_t} \right)$$

$$\forall l, \quad Q^{il}_t = Q^{il}_{t-1} \left(1 + \frac{P^i_t - \widehat{P}^i_t}{\widehat{P}^i_t} \right). \tag{9}$$

Using (7) into (9), we deduce the new quantities:

$$\forall l, \quad Q^{il}_t = Q^{il}_{t-1} \left(1 + \frac{\min \left(\widehat{P}^i_t + \widehat{R}^i_t - r \left(\widehat{D}^i_t - \widehat{L}^i_t \right), \quad \alpha^i \mathcal{A}^i_t \right) - \widehat{P}^i_t}{\widehat{P}^i_t} \right). \tag{10}$$

4 Taking into account bankruptcy

4.1 Hypothesis and definitions

Set of defaulting banks. A time t, if the capital of a non-empty set of banks to drop below a given threshold, those banks declare bankruptcy at time t. Let $\mathcal{D}_t$ be the set of banks that declare bankruptcy at time t. Thus :

$$j \in \mathcal{D}_t \Leftrightarrow \{\widehat{E}^j_t \leq \bar{E}^j\} \cap \{E^j_{t-1} > \bar{E}^j\}. \tag{11}$$

We also define the set of banks that have defaulted up to time t:

$$\mathcal{D}_{0:t} = \bigcup_{s=0}^{t} \mathcal{D}_s.$$

Symmetrically, we denote by $\overline{\mathcal{D}_t}$ the complementary in the set of banks of $\mathcal{D}_t$. We use the same notation for the complementary of $\mathcal{D}_{0:t}$ which we shall then denote by $\overline{\mathcal{D}_{0:t}}$.

Leverage regulatory threshold. In order to analyze the effect of a maximum leverage ratio enforced by the regulator, we can decide moreover that any bank which does not satisfy the leverage regulatory threshold is liquidated as a prudential measure. Mathematically let us introduce the leverage regulatory threshold λ^*, and at all periods banks must satisfy:

$$\frac{\mathcal{L}_t^i}{E_t^i} \leq \lambda^*.$$

This implies a modified definition of the set of defaulting banks:

$$j \in \mathcal{D}_t \Leftrightarrow \left\{ \{E_t^j \leq \bar{E}^j\} \cap \left(\frac{\mathcal{L}_t^j}{E_t^j} > \lambda^* \right) \right\} \cup \left\{ \{E_{t-1}^j > \bar{E}^j\} \cap \left(\frac{\mathcal{L}_{t-1}^j}{E_{t-1}^j} \leq \lambda^* \right) \right\}. \tag{12}$$

Proceedings from liquidation and claim coefficient. Let us firstly introduce two quantities that we will use across this section.

- We define the proceedings from liquidation π_t^j . This is the cash liquidation value of j's balance sheet after it declares default and is liquidated at time t
- We define the claim coefficient of creditor i on bank j by:

$$\Psi_t^{ij} = \frac{\widehat{L}_t^{ij}}{\sum_{s=1}^{n} \widehat{L}_t^{sj}}.$$

In this section, we will present two possible procedures to determine π_t^j : internal settlement in section 4.2 and intervention of a third party in section 4.3. Firstly we need to introduce some considerations of the price impact of fire sales.

Impact of fire sale on risky assets' valuation. When a portfolio is sold in a fire sale context, an important volume is sold and the selling is done in the urgency. It is as a consequence realistic to add a fire sale impact to the valuation of the risky assets sold. There are two dimensions to this impact:

- When banks are liquidated, there is a panic on the markets and since they liquidate their asset, the price of the risky assets at a given period should be locally dependant on the volume of assets liquidated. We model this short term price impact by specifying the fire sale coefficient ξ as a function of the fraction of all risky assets in the system that are being liquidated. Denoting by $0 < v_t < 1$ this fraction, we define ξ_t as:

$$\xi_t = \exp\left(-\theta v_t\right)$$

  Considering θ to be a market sensibility parameter, the bigger it is, the faster the market will react to selling of high volumes. One simple way to specify θ is to express it as the negative of the logarithm of a given ξ_{min} which is the minimal value that we wish ξ_t to attain — in the case $v_t = 1$. For instance if we want the fire sale coefficient to be bounded by 0.2 we can take $\theta = -\ln(0.2)$, which is in fact equivalent to taking:

$$\xi_t = 0.2^{v_t}$$

- We assume that the liquidation of an important volume of a given asset does not have a long term price impact: the price impact is memory-less.[1]

As the reader may wonder, we compute the fraction v_t as the fraction on the total initial quantities, not the remaining total quantities at time t. The underlying interpretation is that the risky assets are sold to a "rest of the world" — markets that are outside of the network. As a consequence the normalization by the total initial quantities is in a way arbitrary only to ensure that we have a fraction that stays within $[0, 1]$. However the absence of adaptive normalization — normalization by the total quantities at time t — reflects the fact that since the risky assets are traded all over the world, and not only from our network to the rest of the world, the quantity of risky assets remaining in the network should not have a direct effect on the price impact of fire-sale from the network to the world.

4.2 Internal settlement

In the internal settlement case, the loans of a defaulting banks j are redistributed to its creditor proportionally to their claim on j.

1. In $t-1$ at stage 2:
   - if $\widehat{E}^j_{t-1} \leq \bar{E}^j$, j declares default. Which triggers the following liquidation steps.
   - π^j_{t-1} is computed. Portfolio is sold according to its valuation with a discount coefficient $0 < \xi_{t-1} < 1$ applied due to fire-sale. Reserves are included.

$$\pi^j_{t-1} = \xi_{t-1}\widehat{P}^j_{t-1} + \widehat{R}^j_{t-1}.$$

[1]Introducing a long term impact on the prices (a lagged effect) would be an interesting add-in to our model.

2. In t at stage 1, the default becomes public information which brings about the following modifications:

   - j's loans are added to the loans of the creditors of j proportionally to Ψ^{ij}_{t-1}:

   $$\forall i \notin \mathcal{D}_{0:t-1}, \quad \forall k \notin \mathcal{D}_{0:t-1}, \quad \widehat{L}^{ik}_t = L^{ik}_{t-1} + \Psi^{ij}_{t-1} L^{jk}_{t-1}.$$

   - The loans matrix is modified to account for j's default:

   $$\forall i, \quad \widehat{L}^{ij}_t = 0$$
   $$\forall k, \quad \widehat{L}^{jk}_t = 0.$$

   The aggregated loans and debts are then computed according to their definition:

   $$\forall i, \quad \widehat{L}^{i}_t = \sum_{s=1}^{n} \widehat{L}^{is}_t$$
   $$\forall i, \quad \widehat{D}^{i}_t = \sum_{s=1}^{n} \widehat{L}^{si}_t.$$

   - j's balance sheet quantities are set to zero :

   $$\left(E^j_t, D^j_t, R^j_t, P^j_t, L^j_t, Q^j_t\right) = \left(0, 0, 0, 0, 0, 0_{\mathbb{R}^m}\right).$$

   - Proceedings of liquidation are distributed to the creditors of j proportionally to their claim on j. This implies a modified version of the equation (3):

   $$\widehat{R}^i_t = R^i_{t-1} + r\left(\widehat{L}^i_t - \widehat{D}^i_t\right) + \Psi^{ij}_{t-1}\pi^j_{t-1}.$$

4.3 Introduction of a third party

Introduction of the liquidator. Another option is to introduce a special actor in the system which we call the liquidator. It only intervenes when a bank is liquidated and cannot go bankrupt (it has infinite reserves). Although for reason that will become clear, the liquidator must be integrated to the graph. We shall as a consequence give it a special index: 0.

The balance sheet of the liquidator has a special format:

Assets ($\mathcal{A}^0_t$)	Liabilities ($\mathcal{L}^0_t$)
Reserves ($R^0_t =$ «∞») Inter-bank loans (L^0_t) (Portfolio ($P^0_t = 0$))	Equity (E^0_t) (Debts ($D^0_t = 0$))

Liquidation with the liquidator. When a bank j goes bankrupt at time $t-1$, the liquidator buys the totality of the bankrupt bank's loans to other banks with a discount $0 < \zeta < 1$. Although, we have to be careful since the bankrupt bank in question may have lent to other defaulting bank. The liquidator does not buy those value-less loans. In all other matters, the procedure is similar to the previous one:

1. In $t-1$ at stage 2:

   - if $\widehat{E}^j_{t-1} \leq \bar{E}^j$, j declares default. Which triggers the following liquidation steps.
   - π^j_{t-1} is computed. Portfolio is sold according to its valuation with a discount coefficient $0 < \xi_{t-1} < 1$ applied due to fire-sale. Reserves are included. Finally cash from the loans sold to the liquidator are incorporated:

$$\pi^j_{t-1} = \xi_{t-1}\widehat{P}^j_{t-1} + \widehat{R}^j_{t-1} + \zeta \sum_{k \in \overline{\mathcal{D}_{t-1}}} \widehat{L}^{jk}_{t-1}.$$

2. In t at stage 1, the default becomes public information which brings about the following modifications:

   - The liquidator's balance sheets is updated to incorporate the loans buy-outs (bank j's debtor now owns money to the liquidator):

$$\begin{aligned}
\widehat{R}^0_t &= \widehat{R}^0_{t-1} - \zeta \sum_{k \in \overline{\mathcal{D}_{t-1}}} L^{jk}_{t-1} \\
\widehat{L}^0_t &= L^0_{t-1} + \sum_{k \in \overline{\mathcal{D}_{t-1}}} L^{jk}_{t-1} \\
\widehat{E}^0_t &= E^0_{t-1} + (1-\zeta) \sum_{k \in \overline{\mathcal{D}_{t-1}}} L^{jk}_{t-1} \\
\forall k \in \overline{\mathcal{D}_{t-1}}, \quad \widehat{L}^{0k}_t &= L^{0k}_{t-1} + L^{jk}_{t-1}.
\end{aligned}$$

   - The loans matrix is modified:

$$\forall i, \quad \widehat{L}^{ij}_t = 0$$
$$\forall k, \quad \widehat{L}^{jk}_t = 0.$$

   The aggregated loans and debts are then computed according to their definition:

$$\forall i, \quad \widehat{L}^i_t = \sum_{s=1}^{n} \widehat{L}^{is}_t$$
$$\forall i, \quad \widehat{D}^i_t = \sum_{s=1}^{n} \widehat{L}^{si}_t.$$

- j's balance sheet quantities are set to zero:

$$\left(E_t^j, D_t^j, R_t^j, P_t^j, L_t^j, Q_t^j\right) = \left(0, 0, 0, 0, 0, 0_{\mathbb{R}^m}\right).$$

- Proceedings of liquidation are distributed to the creditors of j proportionally to their claim on j. This implies a modified version of the equation (3):

$$\widehat{R}_t^i = R_{t-1}^i + r\left(\widehat{L}_t^i - \widehat{D}_t^i\right) + \Psi_{t-1}^{ij}\pi_{t-1}^j.$$

4.4 Analysis of losses

Now that we have introduced the two ways of liquidating, we can decompose the formula for the update of equity in order to better understand the channels of losses. Summing all the proceedings of liquidation that a given bank is entitled to gives us the following update for its equity:

$$\forall i \in [\![0, n]\!], \quad \widehat{E}_t^i = R_{t-1}^i + r\left(\widehat{L}_t^i - \widehat{D}_t^i\right) + \sum_{j \in \mathcal{D}_{t-1}} \Psi_{t-1}^{ij}\pi_{t-1}^j + \widehat{L}_t^i + \widehat{P}_t^i - \widehat{D}_t^i.$$

Developing, we can show that there are three possible sources of losses of equity. Let us consider i given:

$$\widehat{E}_t^i = R_{t-1}^i + r\left(\widehat{L}_t^i - \widehat{D}_t^i\right) + \sum_{j \in \mathcal{D}_{t-1}} \Psi_{t-1}^{ij}\pi_{t-1}^j + L_{t-1}^i + \widehat{L}_t^i - L_{t-1}^i + \widehat{P}_t^i - \widehat{D}_t^i.$$

Using the fact that $\widehat{D}_t^i = D_{t-1}^i$, we can get the following formula:

$$\widehat{E}_t^i = E_{t-1}^i + \underbrace{r\left(\widehat{L}_t^i - D_{t-1}^i\right)}_{\downarrow\ net\ loans\ revenues} + \underbrace{\sum_{j \in \mathcal{D}_{t-1}} \Psi_{t-1}^{ij}\pi_{t-1}^j + \widehat{L}_t^i - L_{t-1}^i}_{defaults} + \underbrace{Q_{t-1}^i\omega_t}_{portfolio}.$$

5 Initialization given a loans structure

We describe in this section how we go about initializing the balance sheets quantities once a weighted and directed graph is given. Obviously this graph entirely characterizes the matrix of loans L, and as a consequence the variables L_0^i and D_0^i for all the banks are fixed. This gives us two degrees of freedom:

- The choice of equities which implicitly determines the amount of liquid assets $\left(P_0^i + R_0^i\right)$.
- The choice of the division of liquid assets between portfolio and reserves.

5.1 Constraints

Although we need to initialize such that the following constraints are satisfied:

- The condition (1) :

$$Q_0^i\mu \geq r\left(D_0^i - L_0^i\right).$$

- The reserve rule (2):

$$R_0^i \geq \max\left(0,\ r\left(D_0^i - L_0^i\right)\right).$$

- A liquid asset postivity rule: Equity must be chosen such that the liquid assets $\left(P_0^i + R_0^i\right)$ are positive, which is equivalent to:

$$E_0^i \geq L_0^i - D_0^i$$

- A leverage objective:

$$\frac{\mathcal{L}_0^i}{E_0^i} \leq \beta^i\lambda^*.$$

where λ^* is a regulatory leverage threshold and β is a behavior parameter for the banks. Since $\mathcal{L}_0^i = E_0^i + D_0^i$, the leverage objective can be rewritten as:

$$1 + \frac{D_0^i}{E_0^i} \leq \beta^i\lambda^*.$$

Since $\frac{D_0^i}{E_0^i} \geq 0$, we must have:

$$\beta^i\lambda^* \geq 1 \Leftrightarrow \beta^i \in \left[\frac{1}{\lambda^*}, 1\right].$$

This interval being valid since $\lambda^* \geq 1$.

The leverage condition can be reformulated as:

$$E_0^i \geq \frac{D_0^i}{\lambda^*\beta^i - 1}.$$

Let us define the leverage lower bound on equity by:

$$E_0^{i*} = \frac{D_0^i}{\lambda^*\beta^i - 1}.$$

5.1.1 Rewriting of the first constraint in terms of portfolio

Let us first focus on the first condition since it needs to be transformed to be expressed as a condition on P_0^i. Firstly, as stated earlier, the relative quantities invested in each asset are given for each bank.

Let us denote by $q^i \in \mathbb{R}^m$ the vector of the relative fractions invested in the risky assets for bank i. As a consequence, we have:

$$Q_0^i = p^i q^i$$

Let us also introduce the portfolio size variable $p^i \in \mathbb{R}^+$. Using this variable, we can characterize P_0^i using the following equality:

$$P_0^i = p^i q^{iT} X_0.$$

We normalize the initial values of all assets, meaning that

$$X_0 = x_0 \mathbf{1}_{\mathbb{R}^m}.$$

As a consequence,

$$P_0^i = p^i q^{iT} x_0 \mathbf{1}_{\mathbb{R}^m} = x_0 p^i$$

Now, we introduce the variable:

$$\tilde{\mu}^i = q^{iT} \mu.$$

We have:

$$p^i \tilde{\mu}^i = Q_0^i \mu$$

and thus:

$$P_0^i \tilde{\mu}^i = x_0 Q_0^i \mu$$

which implies:

$$P_0^i \frac{\tilde{\mu}^i}{x_0} = Q_0^i \mu.$$

We can as a consequence rewrite the condition (1) in terms of portfolio as:

$$P_0^i \frac{\tilde{\mu}^i}{x_0} \geq r \left(D_0^i - L_0^i \right).$$

5.2 Choosing equities

We now want to choose equities such large enough such that the constraints can be satisfied.

Conditions (1) and (2) can be written respectively as:

$$P_0^i \geq \frac{x_0}{\tilde{\mu}^i} r \left(D_0^i - L_0^i\right)$$
$$R_0^i \geq r \left(D_0^i - L_0^i\right).$$

Summing the two, we get:

$$P_0^i + R_0^i \geq r \left(D_0^i - L_0^i\right) \left(\frac{x_0}{\tilde{\mu}^i} + 1\right).$$

Adding L_0^i and subtracting D_0^i on both sides we get:

$$P_0^i + R_0^i + L_0^i - D_0^i \geq r \left(D_0^i - L_0^i\right) \left(\frac{x_0}{\tilde{\mu}^i} + 1\right) + L_0^i - D_0^i$$

which is equivalent to:

$$E_0^i \geq \left(D_0^i - L_0^i\right) \left(r \frac{x_0}{\tilde{\mu}^i} + r - 1\right).$$

Let us define then the quantity:

$$\tilde{E}_0^i = \left(D_0^i - L_0^i\right) \left(r \frac{x_0}{\tilde{\mu}^i} + r - 1\right).$$

We must then choose the initial equity such that it implies enough liquid asset via the above condition and such that it also respects the leverage objective. To finish with, the implied sum of liquid assets must be positive. This can be stated synthetically using the following inequality:

$$E_0^i \geq \max\left(\tilde{E}_0^i, E_0^{i*}, L_0^i - D_0^i\right).$$

5.3 Defining portfolio and reserves

Once the equity has been chosen, the sum $P_0^i + R_0^i$ is implicitly also chosen since $P_0^i + R_0^i = E_0^i - L_0^i + D_0^i$

We can then choose P_0^i and R_0^i solving the following optimization problem:

$$\max P_0^i$$

s.t.

$$P_0^i \le \alpha^i \mathcal{A}_t^i$$
$$R_0^i \ge r\left(D_0^i - L_0^i\right)$$
$$P_0^i + R_0^i = E_0^i - L_0^i + D_0^i$$
$$P_0^i \ge \frac{x_0}{\tilde{\mu}^i} r\left(D_0^i - L_0^i\right)$$

In order to avoid non feasibility, we must ensure when choosing α_i that:

$$\alpha_i \mathcal{A}_0^i \ge \frac{x_0}{\tilde{\mu}^i} r\left(D_0^i - L_0^i\right).$$

Which is equivalent to:

$$\alpha_i \ge \frac{x_0 r\left(D_0^i - L_0^i\right)}{\mathcal{A}_0^i \tilde{\mu}^i}.$$

If α_i is chosen this way, we can apply the same resolution as in the financial choice problem. We have chosen the equity such that it is large enough to satisfy both conditions ((1) and (2))and we maximize the value of the portfolio thus the choice of α_i described above alone ensures that the above problem and the financial choice problem in $t = 0$ are equivalent :

$$\max P_0^i$$

s.t.

$$P_0^i \le \alpha^i \mathcal{A}_t^i$$
$$R_0^i \ge r\left(D_0^i - L_0^i\right)$$
$$P_0^i + R_0^i = E_0^i - L_0^i + D_0^i.$$

Only the equality constraint is different. Taking this into account yields the solution:

$$P_0^i = \min\left(E_0^i - L_0^i + D_0^i - r\left(D_0^i - L_0^i\right), \quad \alpha^i \mathcal{A}_0^i\right).$$

Factorizing by $(D_0^i - L_0^i)$:

$$P_0^i = \min\left(E_0^i + (1-r)\left(D_0^i - L_0^i\right), \quad \alpha^i \mathcal{A}_0^i\right).$$

We deduce the reserves directly from there:

$$R_0^i = E_0^i - L_0^i + D_0^i - P_0^i$$

Since :

$$p^i = \frac{P_0^i}{x_0},$$

and :

$$Q_0^i = p^i q^i,$$

We can easily deduce the initial quantities:

$$Q_0^i = \frac{P_0^i}{x_0} q^i.$$

6 Simulations

We study here banks network in which we initialize the banks' balance sheets such that they are as similar as possible — the balance sheets follow the same distribution. As a consequence, we do not wish to study balance sheet heterogeneity effects for this first set of simulations.

We are going to focus on three indicators for now:

- Cumulative number of defaults. For a given period t this corresponds to the number of banks that have defaulted up to t.
- Fraction of the initial value of the network lost because of defaults up to time t. The initial value of the network corresponds to the sum of the assets of all banks in the network. The value lost because of defaults is the sum of the differences between the losses from defaulting loans minus the proceedings of liquidation. Thus the fraction of the initial value lost because of defaults up to time t is the the sum of the values lost because of defaults up to time t divided by the initial value.
- The probability that 25% of the network has already defaulted at a given period.

6.1 Methodology

6.1.1 Erdős-Réyni graphs

We will carry out our study using the simplest form of random graphs: Erdős-Réyni graphs (**(year?)**). As a reminder, each pair of nodes is linked with a probability p^{ER}, all those edges draws being independent. Thus the connectivity of the graph is controlled by this parameter only. Once such graph is drawn, we know who is connected to who.

We now determine the directions of the previously drawn links. We arbitrarily fix a base direction for the edges (the node in the edge which have the lower index in the nodes pair is put first). We then draw a sign $\{-1, +1\}$ according to a Bernoulli of parameter p^S for each edge. If the sign is -1 the first node of the edge owes money to the second and equivalently the second nodes lent money to the first. We will stick to $p^S = 0.5$ for now.

For simplicity's sake, the weights on the edges are all the same, we call this nominal value l. Although in order to keep the balance sheets comparable for different sets of p^{ER} parameters, we must apply a normalization to l. If we do not, a graph with a high p^{ER} will result mechanically in bigger balance-sheets. Since the mean of the number of edges of Erdős-Réyni graphs of parameter p^{ER} is $(n-1)\,p^{ER}$ and since we only wish to change the parameter p^{ER}, simply dividing l by p^{ER} is a natural normalization.

6.1.2 Random draws and averaging

As a consequence, we have two sources of randomness in this economy for each simulation run.

1. The first—which we will call initial conditions randomness—comes from the initialization of the graph and the balance sheets, it is explained in the previous subsection (6.1.1)

2. The second—which we will call sampled prices randomness—comes from the draw of a price trajectory according to a discrete additive Gaussian random walk (2.2.1).

We will present several types of results. At first, we will draw only one price trajectory and average over several initial conditions draws to get an idea of the reactions brought about by prices events (big drops in risky assets' values or risky assets defaults). Then in order to obtain more robust results, we will average over pairs of initial conditions draws and prices trajectories draws, we call such pairs full draws.

6.1.3 Parameters and context

We keep the parameters simple for our first set of simulations. We work with $n = 100$ banks, $T = 1000$ periods, and $m = 2$ risky assets with each bank investing equally in those assets—maximum portfolio diversification.

We set the mean and standard deviation parameters for both risky assets to be $\mu = 0.01$ and $\sigma = 0.3$—except in some of the prices scenarios for subsection 6.2 where we change those parameter so as to obtain some stereotypical scenarios. The annual interest rate is set to $r_{annual} = 0.05$, we take thus the daily compounded version for the daily interest rate: $r = (1 + r_{annual})^{\frac{1}{365}}$.

We fix the non normalized loans nominal value to 1000, which means that for instance in an Erdős-Réyni graph with $p^{ER} = 0.1$, the face value of all loans will be $\frac{1000}{0.1} = 10000$.

We set the α behavior parameter to be the same for all banks with $\alpha = 0.25$ meaning that if possible, banks will chose their portfolios and reserves such that the portfolio amounts to a quarter of their assets.

The amount of deposits is set at $\bar{E} = 5000$ for every bank meaning that a bank which equity goes under 5000 declares bankruptcy.

We specify the price impact function as $\xi_t = 0.2^{v_t}$. So as to give an idea to the reader, the fire sale function is plotted in the appendix in Figure 10

We will test several values for the regulatory leverage λ^* in what follows. Although, in this setting we do not enforce leverage control at all periods—banks which leverage is above λ^* are thus not forced to declare bankruptcy—we thus use the definition 11 of a defaulting bank rather than the definition 12. In other words, for now we only use λ^* as a control parameter for the initial balance sheets, meaning that at $t = 0$ all banks must have a leverage inferior to λ^*. The procedure for such initialization is described in detail in 5.

We set the parameter $\beta = 1$ for all banks which implies that they are all exactly at the leverage objective upon initialization.

Remark: this choice of β would not be smart if we were to enforce leverage control at all periods, since all the banks that would encounter the slightest loss in the first period would break the leverage threshold and be forced to declare bankruptcy.

6.2 The impact of asset price dynamics.

In this subsection we study the reaction of Erdős-Réyni graphs with different p^{ER} to a given price scenario. We choose three scenarios — one with an asset decreasing slowly by steadily and the other remaining around its initial price, one catastrophic scenario in which one of the risky assets quickly touches zero and one ideal scenario in which both assets' price increases. We set the leverage to $\lambda^* = 5$ and we average the cumulative defaults over 1000 initial condition draws.

6.2.1 A slowly decreasing price scenario

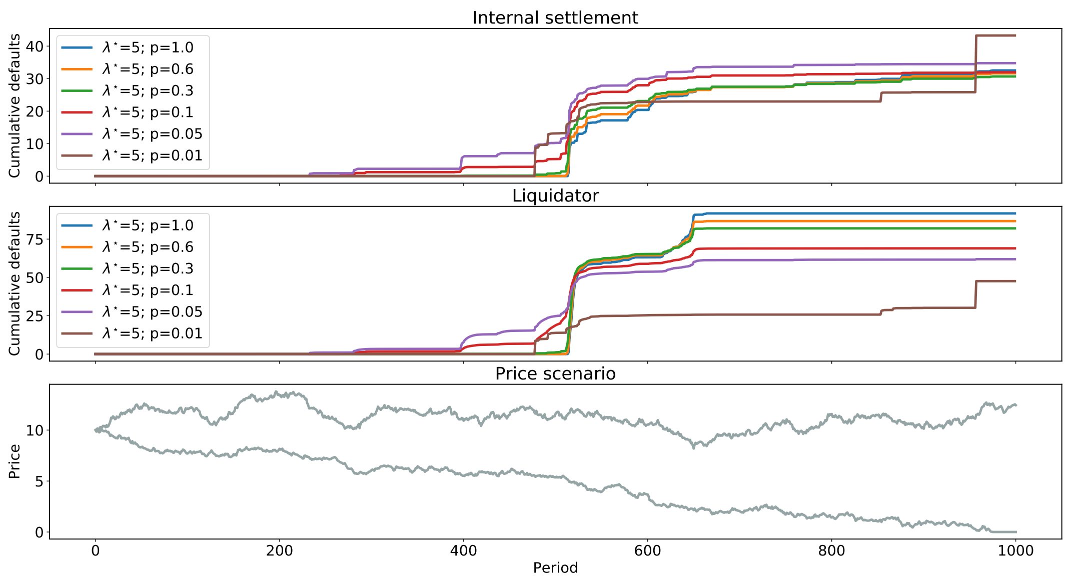

Figure 1: Averaged over 1000 initial conditions

Looking at Figure 1 we observe that for both procedure of liquidation, the least connected graphs — $p^{ER} = 0.01$, $p^{ER} = 0.05$ and $p^{ER} = 0.1$—undergo defaults sooner than the other more connected graphs — $p^{ER} = 0.3$, $p^{ER} = 0.6$ and $p^{ER} = 1$. However the increase in defaults is consequently less abrupt for the least connected graph whereas the first wave of defaults hits the more connected graphs suddenly pushing half of the network into default in just a few periods. One can draw a parallel here with the *robust yet fragile* property of a connected financial network introduced by Acemoglu et al. 2015: a densely connected financial network is more resilient to small shock, but conditionally on a shock hitting hard enough to bring about defaults, the event is very likely to be non-local — a *systemic* crisis. On the other hand the less connected graphs seems more sensitive to small shocks but lack nevertheless the connectivity to be subject — at least for very small connectivity parameters — to abrupt systemic events.

Apart from that, the two liquidation procedures seem to create very different outcomes. With the internal settlement procedure since between half and the entirety of the network have defaulted after 1000 periods. The effect of connectivity also seems different between the two procedures since in the internal settlement case, the graph reaching the highest number of defaults are the least connected ones whereas in the case with the liquidator, more connectivity seems to mean more defaults.

However defaults are not a fatality, if both assets' price increase, we do not observe any defaults which is reassuring for the validity of our model.

6.2.2 Extreme scenarios

Figure 2 shows one of the most terrible cases as one of the assets prices reaches its zero limit quite rapidly which brings about fierce and fast contagion.

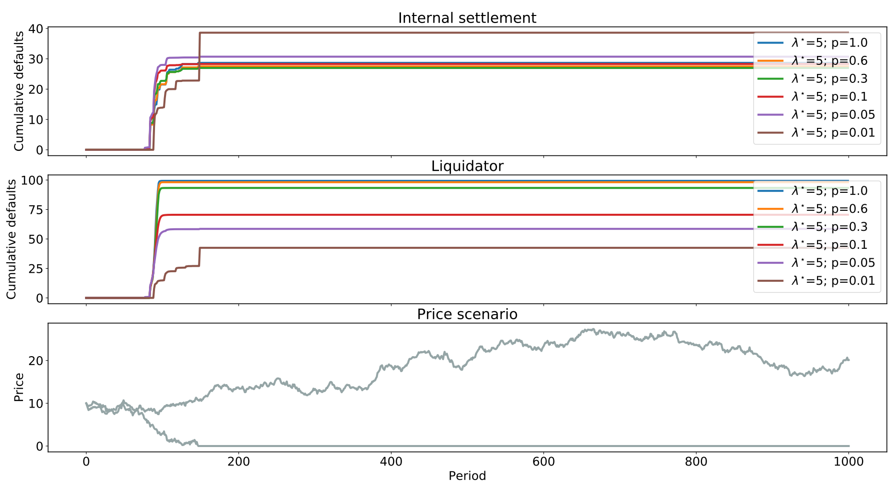

Figure 2: Averaged over 1000 initial conditions

However defaults are not a fatality: if both assets' prices increase, we do not observe any defaults which is reassuring for the validity of our model (see Figure 3).

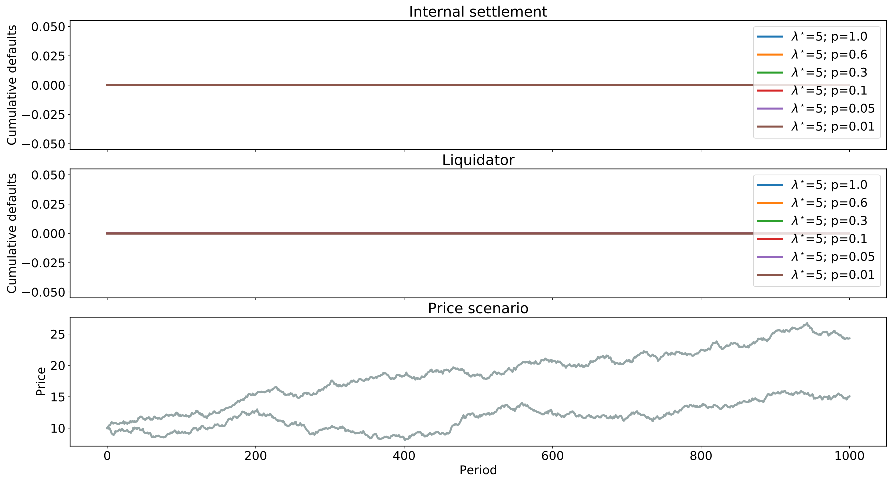

Figure 3: Averaged over 1000 initial conditions

6.3 Impact of connectedness (Fixed λ^*)

We will now average over 4000 full draws (initial conditions and prices) so as to obtain more statistically robust results.

6.3.1 Rewiring

Looking at Figure 4, the number of defaults seems to increase in the parameter p^{ER} from 0.01 to 0.05 where it reaches a maximum and then it decreases with p^{ER} from 0.05 to 1.0. One possible interpretation is that up to a critical point ($p^{ER} = 0.05$) more connectivity means more opportunities for contagion and beyond that critical point more connectivity means a diversification of counterparts reducing potential contagion. To finish with, we are still at a "reasonable" average number of defaults, since roughly one sixth of the network defaults after 1000 periods when p^{ER} is at its critical value. Finally the dynamics of connected networks is markedly different: some curves cross each other. Over time networks with an initial low level of connected ness and lower default rates may eventually generate higher rates.

If we look at the impact on cumulative value lost because of defaults (Figure 5), we find it to be very similar to Figure 4. This tells us that measuring contagion in term of fraction of the initial value lost because of defaults leads us to the same conclusions as measuring it in absolute number of defaults.

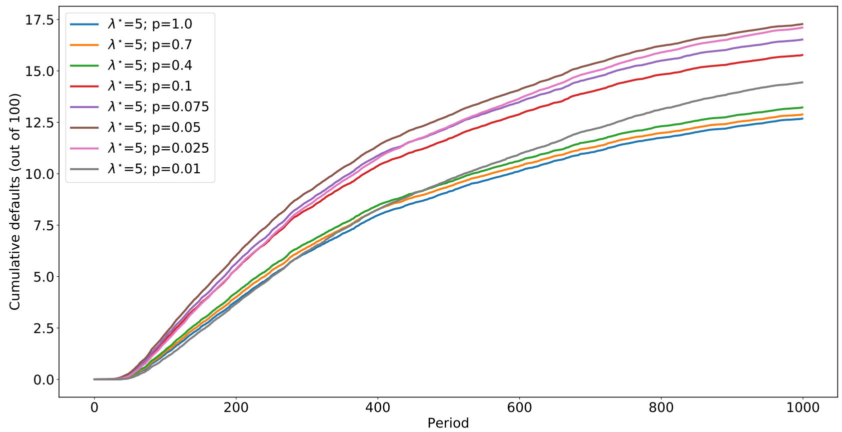

Figure 4: Internal settlement case - Averaged over 4000 full draws

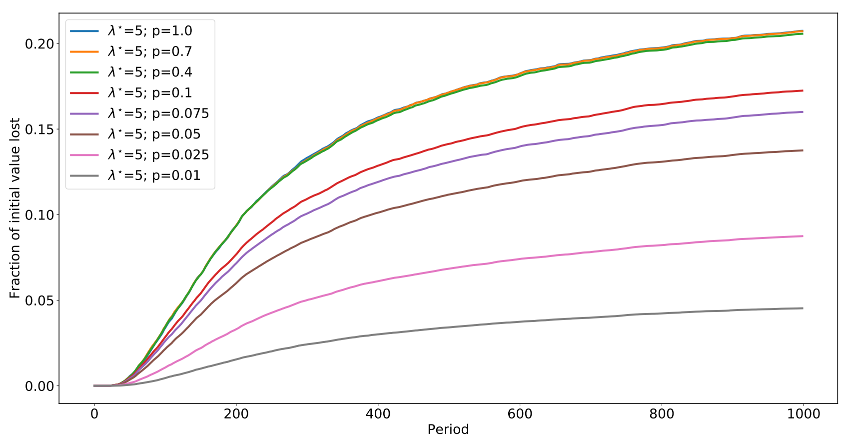

Figure 5: Internal settlement case - Averaged over 4000 full draws

6.3.2 Liquidator

As opposed to the internal settlement liquidation case, Figure 6 shows that, when the liquidation procedure involves the liquidator, the ratio of defaults tends to be more larger for any value of connectedness. There is significantly more defaults using this procedure of liquidation that the previous one. For example, after 1000 periods for $p^{ER} = 0.4$, one third of the network has defaulted with the liquidator and 13% with the previous procedure. This is puzzling but it suggests that the windfall effect of the internal settlement procedure is beneficial as far as contagion is concerned. Indeed it functions as giving more capacity to resist contagion to the lucky banks having borrowed to defaulting ones: the dire consequences of default is concentrated on the lending banks. Some banks are weakened, some are actually strengthened ; on the whole this acts as a protection device, a variety of a containement mechanism.

An increase in more connectiveness on the whole means more opportunities for contagion but it is less systematic than in the internal settlement procedure: we see a slight improvement of $p^{ER} = 0.6$ over $p^{ER} = 0.4$ and of $p^{ER} = 1.0$ over $p^{ER} = 0.6$.

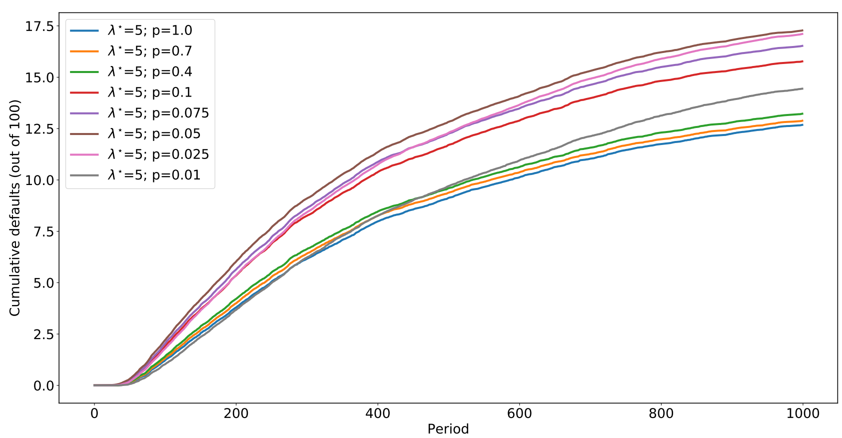

Figure 6: Liquidator case - Averaged over 4000 full draws

The same remarks can almost directly be applied to Figure 7 — losses because of defaults increase with p^{ER}. We however do not observe the same improvement from $p^{ER} = 0.4$ to $p^{ER} = 1.0$ looking at the losses. Lastly, we highlight that in absolute value we reach a maximum of more than 30% of losses in the liquidator case as opposed to 17.5% in the internal settlement case.

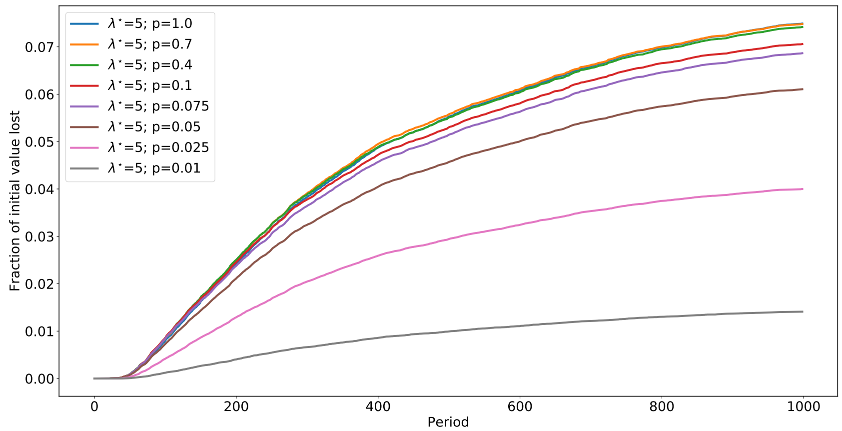

Figure 7: Liquidator case - Averaged over 4000 full draws

6.4 Impact of initial prudential regulation (Effect of λ^*)

We keep the same procedure as before — average over 4000 full draws—but we will now also test several values for λ^*. In order to be able to visualize in 3 dimensions the joint effect of p^{ER} and λ^*, we give ourselves a time horizon of $T = 400$—in the previous graphs, it seems that the increase in defaults and losses is slower after period 400 than after. We then evaluate empirically at each period the probability that 25% of the banks in the network have already defaulted.

6.4.1 Rewiring

Looking at Figure 8, we observe the same type of conclusion for the effect of p^{ER} as in 6.3. As the positive effects of diversification are less explicit in our model, a deeper inquiry is necessary (maybe using other parameters so as to disentangle this effect and see if it is a real finding or rather an artifact). In all cases, such results have a natural interpretation: more connectivity in a graph implies a higher number of potential hits but also reduces the strength of those hits. For smaller values of p^{ER}, the first effect dominates up to a certain point and then the second effect dominates.

The effect of λ^* is less ambiguous and monotone. As one could have anticipated, more leverage implies a higher probability of having already lost one fourth of the network at period 400. Moreover, given p^{ER}, this probability seems to be linearly increasing in λ^*.

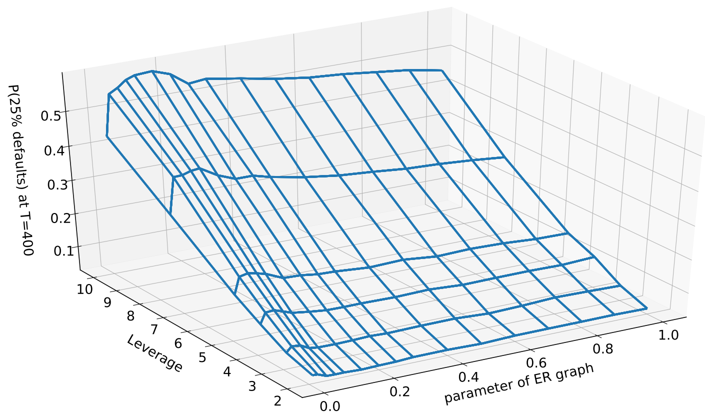

Figure 8: Internal settlement case - Averaged over 4000 full draws

6.4.2 Liquidator

Looking at Figure 9, we do not observe ambiguous effects of diversification when using the liquidation with a liquidator procedure, as opposed to Figure 8. Here more diversification only means the opportunity for more hits. Regarding leverage though, still comparing Figures 8 and 9, the effect of λ^* is similar in both procedures of liquidation.

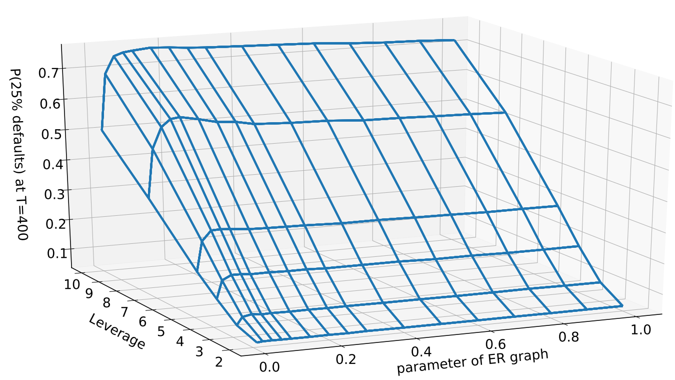

Figure 9: Liquidator - Averaged over 4000 full draws

6.5 Impact of firesale coefficient

As expected the higher the firesale coefficient, the larger the fraction of asset liquidated: this coefficient is an important factor in the fragility of the financial network.

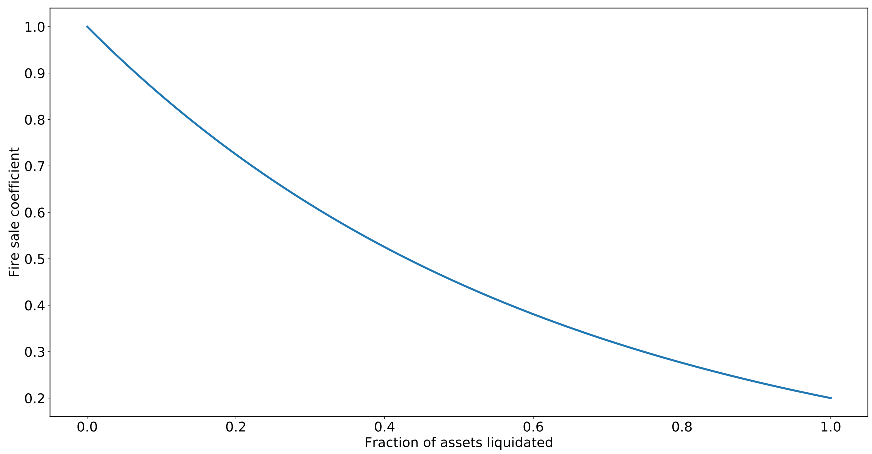

Figure 10: Firesale function

6.6 Network deformation

It is worth studying how do the networks and balance sheets quantities in the network deform typically through time. Here we present preliminary results, acknowledging that more is to be done on the subject.

The simplest way to measure the deformation of the network through time is the "average degree": the average of the sum of in and out degrees for each bank. We use the same simulations as previously (4000 full draws for each value of the parameters). The result for the liquidator's case can be seen in Figure 11. The result is not too surprising and in coherence with the number of defaults from Figure 6, as connections with defaulting banks are broken. The degree of connectedness tend to decrease over time, likely as the result of the disapperance of banks. This is mildly so however: for lowly connected networks the average degree of connectedness hardly moves. What is surprising though is that the curves do not cross: if a network is originally more connected than another one, it remains so despite the different sequence of defaults and the fact that the proportion of defaults tends to be higher for originally more connected networks.

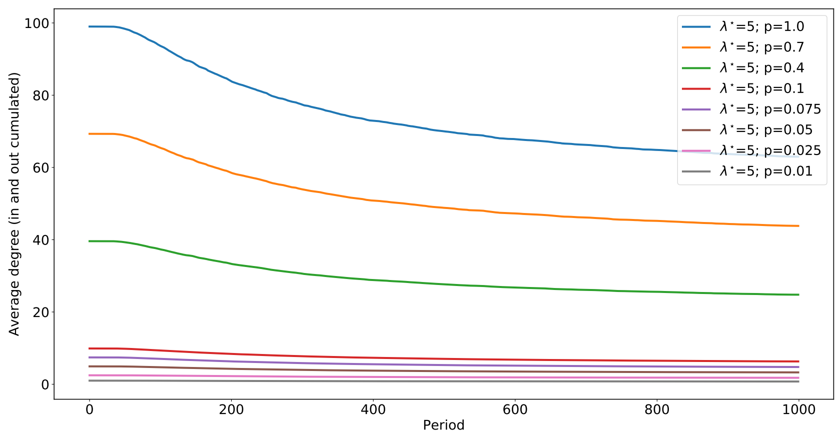

Figure 11: Liquidator - Averaged over 4000 full draws

6.7 Impact of contagion on asset distribution.

We also look at the distributions of different balance-sheet quantities of interest. However, if we want to be able to observe the results graphically, we necessarily need to limit our study to a single set of parameters $\left(p^{ER}, \lambda^*\right)$. We arbitrarily chose for our partial analysis $p^{ER} = 0.3$ and $\lambda^* = 5$.

Let us first look at the average distribution of assets across our 4000 full draws in Figure 12. We observe that the distribution of assets is very piked at first and it then widens

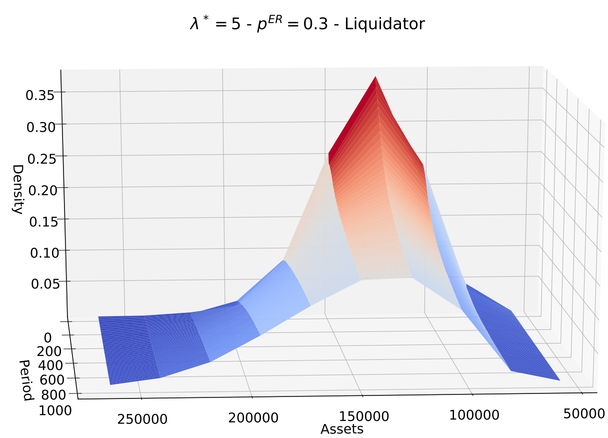

Figure 12: Liquidator - Averaged over 4000 full draws

through time. However it does not do so symmetrically since the distribution becomes skewed thus including banks that are significantly bigger than the others at the end.

We observe the same kind of asymmetric deformation for the distribution of equities, again in the case of liquidator, in figure 13.

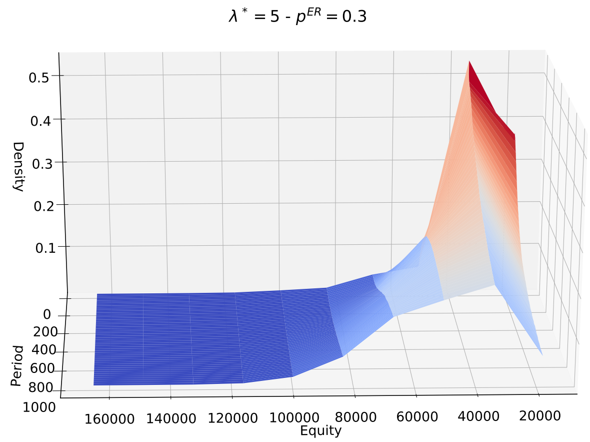

Figure 13: Liquidator - Averaged over 4000 full draws

Another interesting quantity to look at is the evolving distribution of the leverage ratio. Assuming that banks all start out at a fixed regulatory leverage λ^*, this ratio may fluctuate afterwards freely (no prudential dynamic policy is enforced on leverage). Looking at Figure 14, indeed all the banks start out at λ^*, however it seems that the distribution converges to a distribution centered on $\lambda = 3$. If we had time, it would have been interesting to test more seriously if a leverage distribution equilibrium existed and if so, if it were independent of the initial value of λ^*.

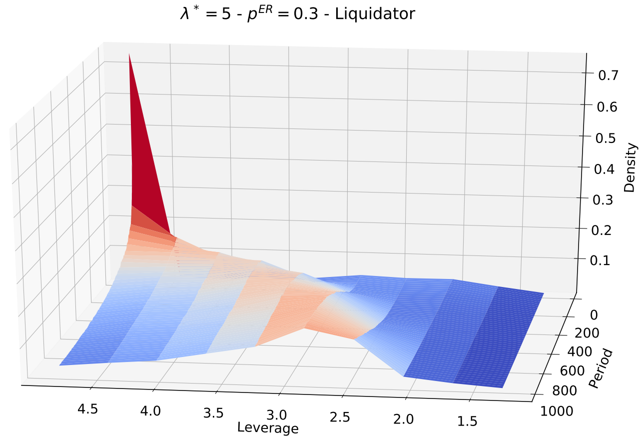

Figure 14: Liquidator - Averaged over 4000 full draws

7 Conclusion

We study in this paper the dynamics of a bank network with a focus on contagion, deformation and convergence of the graph's final structure. We have for now studied indicators mostly regarding banks such as cumulative number of defaults, value lost because of defaults or probability of a given proportion of the network having defaulted at a given period. However we also computed graph indicators such as average degrees, and we will also include examples of network deformations.

Even though we are still at an exploration stage, the simulations yield promising results. The two schemes for liquidation imply two very distinct reactions to the parameter of graph connectivity. When the internal settlement procedure is used, even though increasing the connectivity at first induces more contagion, after a critical connectivity level, the network benefits from diversification. When the liquidator procedure is used however, we do not observe any benefit from diversification as the connectivity of the graph only increases its vulnerability. Moreover, both in absolute number of defaulted banks and in fraction of initial value lost because of defaults, the procedure with internal settlements looks better than the one with the liquidator. In all cases, connectivity matters a great deal. Now focusing on leverage, as opposed to connectivity, its impact is similar for both liquidation procedures, and very significant. As one could expect, more leverage implies a more fragile network more prone to contagion, this fragility seemingly increasing linearly with the leverage.

However, now that we have built a dynamic model of bank network, we wish to test several dynamic control policies and assess their performances. This raises mainly two types of questions: when and where should a regulator intervene ? How should it intervene ? To when and where, we could answer: Using centrality measures ? Using leverage ? Using priority planning (Argyris, Scaman and Vayatis 2016)?... To how we could answer: By forcing liquidation of banked deemed too risky ? By absorbing a part of the balance sheet of the bank in difficulty ? By directly injecting equity ?...

Focusing on the effects of heterogeneity in the balance sheets is our second axis of improvements, in particular, in doing so we wish to investigate the "too big to fail" possibility—are there banks that are so big that their default necessary brings about more defaults that the network can handle ? We also wish to study the key players in the graph structure to see if some are "too connected to fail" or if some are hubs (Zenous 2016) more likely to be vectors of defaults propagation.

As a third axis of improvement, we still need to tune or/and change some parameters and characteristics of the model so as to make it more realistic. Firstly, our characterization of price impact is for instance too simple since it remains local in time. A major improvement to our model would be to incorporate a simple market mechanism modelling offer and demand for the risky assets which would enable us to introduce a long term impact of fire sale on the prices of risky assets. Secondly the banks are very passive actors. We could for instance enrich their behavior by introducing a richer financial choice optimization problem. Thirdly, the interest rates are constant across time and the same for all banks. Adding a dynamic interest individual rate that would take into account the level of risk of each bank would certainly be more realistic. Last but not least, giving the possibility to banks to change their loans partners would be a major add-in.

To finish with, our initialization procedure is not realistic. We find hard to believe—and probably so does the reader—that a real bank network can be likened to an Erdős-Réyni graph. There are bound to be enormous heterogeneity in sizes of balance sheets, in connections intensity... As a consequence, having access to real data would highly improve our credibility. We could thus use a real bank network as initial configuration and potentially find results more likely to fit the complexity of the matter we are trying to model.

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