Can Investors Curb Greenwashing?

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Abstract

We show how investors with pro-environmental preferences and who penalize revelations of past environmental controversies impact corporate greenwashing practices. Through a dynamic equilibrium model, we characterize firms' optimal environmental communication, green investments, and greenwashing policies, and we explain the forces driving them. Notably, under a condition that we explicitly characterize, companies greenwash to inflate their environmental score above their fundamental environmental value, with an effort and impact increasing with investors' pro-environmental preferences. However, investment decisions that penalize greenwashing, policies increasing transparency, and environment-related technological innovation contribute to mitigating corporate greenwashing. We provide empirical support for our results.

Keywords: Greenwashing, sustainable finance, asset pricing, ESG investing, impact investing.

JEL codes: G11, G12, G24.

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1 Introduction

As part of its annual screening of company websites, the European Commission focused on greenwashing practices in 2021. In 42% of cases, the authorities "had reason to believe that the [company's] claim may be false or deceptive."[1] This figure suggests that greenwashing, "the practice by which companies claim they are doing more for the environment than they actually are," is extremely widespread, especially since it can be implemented in a multitude of ways and to varying degrees,[2] and because it is still largely unregulated.[3]

The latest developments in the sustainable finance literature help to understand the prevalence of greenwashing. Indeed, because part of the investors have pro-environmental preferences (Riedl and Smeets, 2017) and internalize environment-related financial risks in their investment decisions (Krüger, Sautner, and Starks, 2020), green companies benefit from a lower cost of capital in equilibrium (Pástor, Stambaugh, and Taylor, 2021; Pedersen, Fitzgibbons, and Pomorski, 2021; Zerbib, 2022). In addition, companies' environmental footprints are challenging to measure accurately,[4] measurement methods are not standardized (Berg, Koelbel, and Rigobon, 2022), and companies may communicate about their environmental footprint in an ambiguous manner (Fabrizio and Kim, 2019). Thus, companies have the ability and the incentive to overstate their environmental value with the aim of increasing their environmental score.

By misinforming stakeholders about the environmental impact of companies, greenwash-

[1] https://ec.europa.eu/commission/presscorner/detail/en/ip_21_269

[2] https://futerra-assets.s3.amazonaws.com/documents/The_Greenwash_Guide.pdf

[3] With the notable exception of the European Union, where a draft European directive aimed at banning "Generic environmental claims and other misleading marketing tricks" is in preparation and could come into force in 2026 if passed by member states: https://www.europarl.europa.eu/news/en/press-room/20230918IPR05412/eu-to-ban-greenwashing-and-improve-consumer-information-on-product-durability.

[4] For the basic example of climate change, there are several issues to contend with, such as the accuracy of disclosure for scopes 1 and 2, and the availability of information on scope 3. For other environmental topics, the challenge is often even greater; for example, the calculation of a biodiversity footprint is rudimentary and approximate, given the number of assumptions that rating agencies have to make (Garel, Romec, Sautner, and Wagner, 2024).

ing creates a major obstacle to the ecological transition. Specifically, greenwashing has a negative impact on sustainable investment for two primary reasons: (i) it complicates the evaluation of the environment-related financial risks, and (ii) it reduces sustainable investors' positive impact on the environmental practices of companies by making more challenging the evaluation of their environmental footprints. In this paper, we notably show how sustainable investors may indirectly incentivize companies to practice greenwashing, and how they can directly discourage them from doing so.

We build a dynamic equilibrium model populated by n heterogeneous firms and a representative investor having imperfect information about the environmental values of companies. Each firm has a fundamental environmental value (also referred to as "environmental value"), which it can adjust continuously by investing in green projects at a quadratic cost. However, the investor does not observe companies' environmental values and relies on imperfect environmental scores estimated by a third party such as a rating agency. Companies can influence their environmental scores directly through costly environmental communication, which can be true or deceptive. Deceptive positive ("green") communication consists in increasing the environmental score without increasing the environmental value, creating a spread between the two. However, the environmental score reverts back towards the environmental value over time through the action of two forces: (i) continuously, through the work of the rating agency, and (ii) discontinuously, through events, to which we also refer as "controversies," which instantly and publicly reveal a random fraction of the spread between the score and the environmental value. The occurrence of these events is modeled through a Poisson process. The average rate of discovery of the environmental value through these two forces, which we define as the "revelation intensity," characterizes the degree of quality of the environmental score of each company.

The representative investor has two main features: she can have pro-environmental preferences (e.g., Pástor et al., 2021; Zerbib, 2022) and can penalize the spread between a company's score and its environmental value, when it is revealed by controversies. We also refer to this penalty as a "penalty on revealed misrating" or "*misrating* penalty." This penalization can be interpreted as a readjustment of the environmental score, which is considered insufficiently credible, or as an additional penalty linked to poor corporate governance. It echoes other forms of misconduct penalties (e.g., Egan, Matvos, and Seru, 2022).

The investor allocates her capital among n firms with dynamic mean-variance preferences (e.g., Buffa, Vayanos, and Woolley, 2022) adjusted to reflect pro-environmental preferences and the penalty associated with revealed score inaccuracies. The firms dynamically choose their (i) greening efforts and (ii) communication efforts to minimize the sum of their equilibrium capital costs, greening costs and environmental communication costs. From the optimal communication and greening efforts, we derive the greenwashing strategy of a company, defined as a green communication effort that aims at creating or increasing a positive gap between the environmental score and the environmental value.

Through our baseline model, we document four main results relating to (1) equilibrium expected returns, (2) companies' optimal environmental communication and greening strategies, (3) companies' optimal greenwashing strategy and how investors can curb it, and (4) complementary tools available to policymakers to limit greenwashing and incentivize greening efforts. All the results we obtain are closed-form formulas, allowing us to analyze the underlying effects.

First, we show that the investor's penalization of revealed misrating commands a premium on expected returns, which scales with the strength of the penalty. In addition to the green premium documented by Pástor et al. (2021), Pedersen et al. (2021), and Zerbib (2022), the investor requires higher returns to hold stocks whose environmental score credibility is questionable in light of past controversies.

Second, the optimal greening effort and environmental communication of each company is derived analytically in feedback form as a function of its current environmental score and environmental value. The two environmental strategies of company i jointly serve the purpose of increasing its environmental score without moving it too far from its fundamental environmental value. While these two strategies are perfect substitutes when the investor *only* takes into account pro-environmental preferences or *only* penalizes revealed misrating, they become complementary when the investor combines pro-environmental preferences with a penalty on revealed misrating.

Third, the optimal greenwashing effort can be derived from the explicit expressions of these two strategies, and interpreted as follows. Pushed by the investor's pro-environmental preferences, companies greenwash as long as it is sufficiently cheap to engage in environmental communication relative to greening investments, the asymmetry of information is sufficiently strong, or their rate of time preference is high enough. Through greenwashing, companies try to maintain a certain level of overrating in their environmental score. In particular, the optimal environmental communication effort is countercyclical with respect to the environmental score: the higher the environmental score, the lower the communication effort. The condition for the existence of greenwashing does not depend on the investor's misrating penalty, but this penalty reduces the extent of greenwashing. This penalty is, therefore, a useful tool in the hands of sustainable investors to counterbalance the indirect greenwashing incentive they transmit to companies through their pro-environmental preferences. This penalty also increases companies' incentive to make greening efforts, thereby enabling investors to increase the positive impact they have on companies' environmental practices.

Fourth, we examine two complementary policy instruments for reducing greenwashing: regulations to increase transparency on corporate environmental practices, and support for environmental technological innovation. Increasing the ability of rating agencies to unravel the fundamental environmental value of companies is an instrument that acts as a substitute to investors' misrating penalty, as the two instruments have similar effects that do not add up. Conversely, promoting media and stakeholders' efforts to detect controversies is complementary to this penalty and amplifies its effect. As for environmental technological innovation, it can only reduce greenwashing when it significantly lowers the marginal unit costs of greening compared with those of communication. Thus, maintaining a sustained and pronounced research and development effort to bring down the marginal costs of new green technologies would, in addition to increasing greening efforts, simultaneously help curb corporate greenwashing practices.

What if investors only care about *relative* environmental scores of companies, either because they practice best-in-class investment strategies or because rating agencies standardize scores? This practice introduces interaction between companies' environmental strategies, which try to outperform each other. We formulate an extension to the model, in which the investor normalizes each company's environmental score by the average environmental score in the investment universe. Through a mean field approximation detailed in the Internet Appendix, we solve this game and prove that it admits a unique Nash equilibrium. Analytically, we find that the optimal environmental strategy of a representative company follows a similar pattern to the baseline case. Hence, the qualitative conclusions stated above are robust to the introduction of such an interaction between companies. However, we show that this interaction leads to lower greening, communication and greenwashing efforts than in the baseline case. Indeed, since the company's objective is now to outperform its peers, the incentive for having a high absolute environmental score is weaker. These results suggest that the commonly used cross-sectional normalization of companies' environmental scores by rating agencies and the best-in-class approaches to portfolio selection may have a detrimental impact on the extent of improvement in firms' environmental performance.

We provide empirical evidence supporting the results of our model. Because greenwashing practices are unobservable, we focus on global companies' environmental communication from December 2015 to December 2022, and we document two main results: (i) we show that companies almost structurally engage in *green* (i.e., positive environmental) communication, and (ii) we validate the dynamics of the environmental communication found in our model.

We propose a two-step empirical method for analyzing companies' environmental communication policies and testing their dynamics in cross-section. To do so, we use monthly data from the data provider Covalence, which constructs an environmental reputation score, an environmental controversy score, and an environmental performance score from published news. In the first step, we construct a proxy for the environmental communication score and find that the monthly average flow of environmental communication is positive 98.8% of the time, that is, companies almost structurally engage in green communication (result [i]). In the second step, we provide empirical support for the environmental communication dynamics highlighted by the model (result [ii]). To do so, we make the natural assumption that the fundamental environmental value, which is unknown, is relatively inert at the monthly frequency. Under this assumption, we perform a Within regression of the monthly change in environmental communication on the monthly change in environmental score instrumented by the past environmental score, given the simultaneity issue. Through a number of complementary estimations (different sub-samples, different starting dates, and different environmental sub-scores), we find strong evidence that companies steer their environmental communication in a counter-cyclical way with respect to the evolution of their environmental score, consistent with the effect of the corrective force highlighted above.

Because the academic literature has not documented any structural underestimation of environmental scores, and given the low marginal unit costs of communication compared with those of greening efforts (Bank for International Settlement, 2017) as well as the asymmetry of information that companies enjoy about their fundamental environmental value (Barbalau and Zeni, 2023), the quasi-structural and countercyclical green communication policy of companies suggests that they may engage in greenwashing, at least part of the time, consistent with the results of our model.

Related literature. Our results extend prior research on greenwashing, asset pricing, and impact investing. First, our paper contributes to the nascent financial literature on greenwashing.[5] Corporate greenwashing has increased significantly over the past five years (Gourier and Mathurin, 2024) and is particularly prevalent in cases where companies benefit from information asymmetry (Wu, Zhang, and Xie, 2020). For example, forms of greenwashing have been documented through conflicts of interest between companies and the firms auditing them (Duflo, Greenstone, and Ryan, 2013), as well as, indirectly, when companies sell polluting plants to companies facing weaker environmental pressures without inducing a reduction in overall pollution (Duchin, Gao, and Xu, 2023). However, empirical evidence suggests that investors can contribute to reducing corporate greenwashing: by participating in climate initiatives using the shareholder engagement channel, investors reduce corporate cheap talk on climate issues (Bingler, Kraus, Leippold, and Webersinke, 2023). Yet, asset managers are not exempt from suspicions of greenwashing (Kim and Yoon, 2022), and instances of greenwashing in the news lead to capital outflows from funds marketed as sustainable (Gourier and Mathurin, 2024).[6] We contribute to this literature by developing, to the best of our knowledge, the first theoretical model linking corporate greenwashing to investor pressure, along with a contemporary working paper by Chen (2024). Specifically, we characterize the mechanisms that induce and reduce corporate greenwashing from an asset pricing perspective, and we provide empirical evidence for them.

Chen (2024) addresses a question similar to ours through a theoretical model. However, we differ from this paper as (i) we explicitly characterize equilibrium asset returns and optimal greenwashing strategies, (ii) in a dynamic setup, (iii) allowing for interaction

[5] Besides the financial literature, which we review below, studies in adjacent research fields have addressed the issue of greenwashing from the business ethics standpoint, see for example, Laufer (2003), Walker and Wan (2012), Lyon and Montgomery (2015), Marquis, Toffel, and Zhou (2016).

[6] These results echo the literature on disclosure, which highlights investors' increased demand for transparency (Flammer, 2021; Ilhan, Krueger, Sautner, and Starks, 2023), as well as the literature documenting the divergence between ESG rating providers (Berg et al., 2022), the opacity of data construction (Berg, Fabisik, and Sautner, 2021), and the lack of forward-looking perspective in ESG ratings (Bams and van der Kroft, 2024; van Binsbergen and Brogger, 2024), emphasizing the complex nature of investment decisions based on ESG criteria. In addition, the dynamics of environmental transparency regulations have a significant impact on the ecological transition of companies (Gupta and Starmans, 2024) as well as the GDP when associated with a carbon tax (Frankovic and Kolb, 2024).

among companies to choose their optimal policies, and (iv) providing empirical evidence for our results. In addition, from the model assumptions standpoint, we remain agnostic on the difference in NPV of green and brown projects and we allow investors to selectively penalize companies that greenwash thanks to the advent of controversies. Thus, we reach different conclusions: in Chen (2024), investors' environmental impact decreases with pro-environmental preferences because all companies are penalized by greenwashing, while in our paper, investors' impact increases with these preferences as greenwashing is penalized at the firm level.

We also contribute to the asset pricing literature. Whether for climate (Engle, Giglio, Kelly, Lee, and Stroebel, 2020; Choi, Gao, and Jiang, 2020; Sautner, van Lent, Vilkov, and Zhang, 2023) or biodiversity (Giglio, Kuchler, Stroebel, and Zeng, 2023; Garel et al., 2024; Coqueret, Giroux, and Zerbib, 2024) issues, the pro-environmental preferences of investors (Riedl and Smeets, 2017; Humphrey, Kogan, Sagi, and Starks, 2023) and their expectations of future environmental risks (Krüger et al., 2020; Stroebel and Wurgler, 2021; Hambel, Kraft, and van der Ploeg, 2023) command a green premium that increases the cost of capital of the most polluting companies (Pástor et al., 2021; Pástor, Stambaugh, and Taylor, 2022; Pedersen et al., 2021; Bolton and Kacperczyk, 2021; Zerbib, 2022; De Angelis, Tankov, and Zerbib, 2023; Hsu, Li, and Tsou, 2023; Bolton and Kacperczyk, 2023; Cenedese, Han, and Kacperczyk, 2024).[7] We contribute to the sustainable asset pricing literature by showing that investors' penalties for misratings revealed during environmental controversies command a

[7] The effect of the green premium increases with the inelasticity of the demand function of passive sustainable investors (Cheng, Jondeau, Mojon, and Vayanos, 2023), but is attenuated in the presence of uncertainty (De Angelis et al., 2023; Avramov, Cheng, Lioui, and Tarelli, 2022) as well as when green investors also have green consumption preferences (Sauzet and Zerbib, 2022); it can even be almost zero when the investors' demand function is elastic (Berk and van Binsbergen, 2021). In addition, a green premium driven by non-pecuniary motives may alter equilibrium prices in a suboptimal manner from a climate risk perspective (Goldstein, Kopytov, Shen, and Xiang, 2022). It is noteworthy that the rise in the cost of capital of brown companies has been associated in recent years with an increase in the financial performance of the greenest assets due to an unexpected increase in pro-environmental preferences (Pástor et al., 2022; Ardia, Bluteau, Boudt, and Inghelbrecht, 2023), and hence, in the price impact of these flows towards green assets (Van der Beck, 2023).

risk premium that increases the cost of capital of the companies whose reputations have been tarnished.

Finally, we contribute to the growing literature on impact investing. Even if the green premium induced by sustainable investment increases the cost of capital of the most polluting companies, the incentive to go green for these companies remains limited (De Angelis et al., 2023), and may even have a counter-productive effect by increasing the environmental footprint of polluting companies, which turn to brown projects that generate short-term cash flows (Hartzmark and Shue, 2023). Yet, Favilukis, Garlappi, and Uppal (2023) show that constrained mandates on green investment can significantly influence the allocation of capital across sectors with a negligible impact on the cost of capital. In any case, a number of papers highlight conditions under which investors can increase their impact on the greening of corporate practices: basing investment decisions on aggregate welfare by internalizing the externalities of all firms in the economy (Green and Roth, 2024; Oehmke and Opp, 2024), funding companies that would not have been funded by regular investors otherwise (Green and Roth, 2024), prioritizing investments where search friction is acute (Landier and Lovo, 2023), and holding a brown stock if it has taken corrective action (Edmans, Levit, and Schneemeier, 2023).[8] We contribute to this literature by showing that green investors can have a double impact on corporate practices: indirectly, by encouraging companies to greenwash through their pro-environmental preferences, and directly, by limiting corporate greenwashing and spurring green investments through the penalties they apply when an environmental controversy is revealed.

Outline. This paper is structured as follows. Section 2 introduces an economy populated by companies able to greenwash but exposed to the investor penalty. Section 3 describes

[8]Nevertheless, the limits of impact investing are reflected in investors' willingness to pay for impact (Barber, Morse, and Yasuda, 2021), which is low compared to the willingness to pay to invest in green assets (Bonnefon, Landier, Sastry, and Thesmar, 2022) and, when it exists, does not scale with the level of impact (Heeb, Kölbel, Paetzold, and Zeisberger, 2023).

the equilibrium pricing equation as well as firms' optimal greening, communication, and greenwashing strategies. Section 4 presents an extension of the investor's program with firm interaction and summarizes the main findings in this new setting. Section 5 provides empirical evidence supporting the findings of the model, and Section 6 concludes the paper. In the Internet Appendix, we give a formal definition of the notion of marginal benefits (Internet Appendix Section 1), we provide the study of two limiting cases (Internet Appendix Section 2), we gather all the proofs of the paper in the general case (Internet Appendix Section 3), we present the calibration used for the simulations (Internet Appendix Section 4), we give the proofs of the model extended to the case wherein firms interact in a mean field game (Internet Appendix Section 5), and we give the set of complementary regression tables from the empirical analysis (Internet Appendix Section 6).

2 An equilibrium model with corporate greenwashing

We build a framework based on a dynamic asset pricing model with imperfect information about companies' environmental value, wherein companies and investors interact as part of a Stackelberg game.

Market setting. Our model is inspired by the dynamic asset pricing model of Bouchard, Fukasawa, Herdegen, and Muhle-Karbe (2018), where the volatility matrix of asset prices is exogenous, the expected return vector is determined in equilibrium, and the representative investor maximizes a mean-variance objective. Unlike the above reference, we do not allow for transaction costs, but we introduce additional terms in the investor's objective function to account for non-pecuniary preferences and misrating penalty. On a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with infinite time horizon, we consider a market with n firms, indexed by $i \in \{1, \ldots, n\}$, issuing stocks at date 0, and a representative investor.[9] The price process $S \in \mathbb{R}^n$ is assumed to follow the dynamics

$$dS_t = \mu_t dt + \sigma dB_t, \tag{1}$$

where $\mu_t \in \mathbb{R}^n$ is the vector of expected returns of the assets, which is determined in equilibrium, $\sigma \in \mathbb{R}^{n \times n}$ is the exogenously specified volatility matrix, which is assumed to be constant and nonsingular,[10] and (B_t) is an n-dimensional Brownian motion. The quantity of stocks of each company is normalized to one. In addition to risky assets, the investor can also invest in a risk-free asset, which is assumed to have a zero rate, without loss of generality. In this paper, the i-th component of a vector $h \in \mathbb{R}^n$ is denoted by h^i.

We do not make any assumption about the dividends or earnings of each company: we remain agnostic about the impact of a company's green investment and communication on its earnings and dividends, in line with Gupta and Starmans (2024). This modeling choice also prevents the representative investor from learning about companies' environmental values through the observation of their earnings or dividends, which would be incoherent with the assumption of imperfect information.

Environmental score. The fundamental environmental value of each company i, denoted by V^i, is not observed by the investor. Instead, she observes an environmental score, namely, a public rating provided by a rating agency, which aims to estimate the fundamental environmental value but does not perfectly reflect it due to imperfect information. The environmental score of company i, E^i, depends on the company's fundamental environmental

[9]We consider a representative investor to avoid unnecessary model complexity. The main conclusions remain unchanged in a model with several investors with heterogeneous preferences.

[10]The assumption of a constant exogenous volatility matrix is consistent with what most of the sustainable asset pricing literature has assumed to date (e.g., Pástor et al., 2021; Pedersen et al., 2021; Zerbib, 2022). The exploration of models with endogenous volatility matrix involves significant complexities, which prevent the obtention of closed-form solutions, as we allow for stochastic adapted strategies for companies, as opposed to, for example, De Angelis et al. (2023). We leave this interesting avenue for future research.

value, V^i, and its environmental communication effort, c^i, as follows:

$$dE_t^i = \underbrace{a(V_t^i - E_t^i)dt}_{\text{Rating agency effect}} + \underbrace{(V_{t-}^i - E_{t-}^i)\Theta_t^i dN_t^i}_{\text{Controversy effect}} + \underbrace{c_t^i dt}_{\text{Communication effect}} + \underbrace{zdW_t^i}_{\text{Measurement error}}, \quad E_0^i = q^i, \tag{2a}$$

$$dV_t^i = \underbrace{v_t^i dt}_{\text{Greening effect}}, \quad V_0^i = p^i, \tag{2b}$$

where $a, b, z, q^i, p^i \in \mathbb{R}_+$ are constant deterministic parameters, (N_t^i) is a one-dimensional Poisson process of intensity $\lambda^i \in \mathbb{R}_+^*$, and (W_t^i) a one-dimensional Brownian motion.

The fundamental environmental value of company i, V^i, is determined by its environmental footprint reduction or "greening" effort, v^i. However, since the rating agency does not directly observe greening efforts, the score, E^i, can be influenced by environmental communication, c^i, and measurement noise or error, zW_t^i.[11] Environmental communication and measurement error can both contribute to creating a discrepancy between the environmental score and fundamental environmental value. Notably, we refer to *green (brown) communication* when a company engages in environmental communication that has a positive (negative) impact on its environmental score, that is, when $c_t^i > 0$ ($c_t^i < 0$).[12] These effects are counterbalanced by two mechanisms contributing to reveal its fundamental environmental value. First, continuous efforts of the rating agency create a force pushing the environmental score towards the fundamental environmental value with speed a. Second, controversies related to the environmental quality of the company arise at random times and contribute to reveal-

[11]Berg et al. (2022) estimate that measurement differences explain 56% of ESG scores divergence across ESG rating agencies.

[12]Whether truthful or deceptive, green communication refers to positive environmental communication made by a company to convince that its environmental value is higher than its current environmental score: it can be a pledge on greening targets, environmental reporting, or attractive ways to present its environmental policy when answering rating agencies' questionnaires. Brown communication refers to any communication made by a company that adversely affects its public environmental image. The company might opt to backtrack on a previous environmental commitment, announce the abandonment of an emission reduction target, or disclose information regarding its unexpectedly substantial environmental footprint.

ing its fundamental environmental value. A controversy at time t reveals a random portion $\Theta_t^i \in [0,1]$ of the ongoing *misrating* prior to the occurrence of the controversy $|E_{t-}^i - V_{t-}^i|$.[13] The rating agency may adjust the rating partially, rather than fixing it to be exactly equal to the environmental value, for several reasons. For example, the ESG rating has several pillars, and the controversy may affect only one, or a subset, of these pillars, as explained in Example 1 below. We assume that the proportion of the spread revealed at each controversy follows the beta distribution $B(1, 1/b - 1)$ with $b \leq 1$, and is independent both from the value of the spread and from the process $(N_t^i)_{t\geq 0}$ governing the occurrence of controversies. By convention, when $b = 1$, we take $\Theta_t^i \equiv 1$. The parameter b corresponds to the expected proportion by which the spread is adjusted: $b = \mathbb{E}[\Theta_t^i]$. Controversies are assumed to arise independently from the measurement error, that is, for each company i, N^i and Θ^i are independent from W^i.

Example 1 (Random rating adjustment)**.** Assume that the environmental score is composed of K pillars. For each pillar, and for each company, there is a fundamental value, denoted by $V_t^{i,k}$, and the estimated score, denoted by $E_t^{i,k}$, with $V_t^i = \frac{1}{K}\sum_{k=1}^K V_t^{i,k}$ and $E_t^i = \frac{1}{K}\sum_{k=1}^K E_t^{i,k}$. We denote the spread between the two by $\pi_t^{i,k} = E_t^{i,k} - V_t^{i,k}$. In the event of a controversy, the score for one of the pillars reverts to its fundamental value. Assume that the score for k-th pillar is adjusted. This means that the overall environmental rating adjustment, at a time t when $dN_t^i \neq 0$, satisfies,

$$E_t^i = E_{t-}^i - \frac{1}{K}\pi_{t-}^{i,k} = E_{t-}^i - \frac{\pi_{t-}^{i,k}}{\sum_j \pi_{t-}^{i,j}}(E_{t-}^i - V_{t-}^i).$$

In this example, we can write $\Theta_t^i = \frac{\pi_{t-}^{i,k}}{\sum_j \pi_{t-}^{i,j}}$ when $dN_t^i \neq 0$, and $\Theta_t^i = 0$ otherwise.

Suppose that the rating agency does not know the individual errors $\pi_t^{i,j}$ and that the

[13] We refer to controversies as events that reveal a discrepancy between a company's environmental score and its environmental value. These discrepancies can be positive or negative, in line with the definition of a controversy as "a disagreement or strong debate" (Cambridge Dictionary).

ratio Θ_t^i is independent from $\sum_j \pi_t^{i,j}$ and has the same distribution as the minimum of $K-1$ independent uniform random variables (this is for example the case if $\pi_t^{i,k}$, $k = 1, \dots, K$, are independent exponential), with the convention that $\Theta_t^i \equiv 1$ if $K = 1$. Then,

$$\mathbb{P}[\Theta_t^i \geq u] = (1-u)^{K-1},$$

which is the beta distribution $B(1, K-1)$, with the convention that $B(1, 0)$ is the law of a constant equal to 1. To account for other possible reasons for partial rating adjustment,we replace K with a continuously varying parameter.

Now, we can define the practice of greenwashing, which is a green communication strategy whereby a company oversells its environmental image. Recall that the environmental score, E^i, which is controlled by the communication effort, c^i, aims to estimate the fundamental environmental value, V^i, which is controlled by the greening effort, v^i. The two efforts, c^i and v^i, impact similarly E^i and V^i, respectively, and are measured in the same units.

Definition 1 (Greenwashing)**.** Company i is *greenwashing* at time t if (i) it is overrated, that is, $E_t^i \geq V_t^i$, (ii) its environmental communication is positive, $c_t^i > 0$, and (iii) it communicates more than it abates, $c_t^i > v_t^i$. When company i is greenwashing, its *greenwashing effort* is defined as $c_t^i - v_t^i$.

The first two criteria reflect the fact that a company engages in green communication when it is already overrated in terms of its fundamental environmental value. The third criterion allows us to exclude from the scope of greenwashing cases where a company is genuinely communicating about the launch of a new green project ($c_t^i \leq v_t^i$), even though it is already overrated. Greenwashing is, therefore, any green communication effort that aims at creating or increasing a positive gap between the environmental score and the fundamental environmental value, when the company is accurately rated or already overrated.

Investor's score for environmental misrating. The investor has a preference for informative environmental scores, as she wishes to allocate her capital to green companies based on accurate information. Therefore, in her asset allocation program, she penalizes companies whose environmental scores have proven inaccurate in the past. The investor builds a score M_t^i for company i at time t, based on the environmental score inaccuracies she has observed through past controversies as follows:

$$dM_t^i = -\rho M_t^i dt + (E_t^i - E_{t-}^i)^2 dN_t^i, \quad M_0^i = u^i, \tag{3}$$

with $\rho, u^i \in \mathbb{R}_+$. At each controversy, that is, when $dN_t^i = 1$, the score M^i jumps upwards, according to the square of the revealed misrating, $|E_t^i - E_{t-}^i|$. This score for misrating is quadratic in the environmental score adjustment because the effect of controversies usually induces dramatic and non-linear repricing (see, for example, the impacts of the 2010 British Petroleum, 2015 Volkswagen, and 2015 ExxonMobil controversies on asset prices). When there is no controversy, the score M^i is continuous and decreases at rate $\rho > 0$, as the investor gives more importance to recent controversies than older ones. Note that the misrating score, M^i, is positive.

It should be noted that this specification assumes that the investor penalizes all types of inaccuracies, be they positive or negative. In theory, this assumption is justified by the investor's need for transparency on the fundamental environmental values of companies to improve her capital asset allocation. In practice, as we will show below, the companies' scores are pulled upward by the investor's pro-environmental preferences, and controversies generally drive the scores down toward the companies' environmental values.

Program of the investor. The program of the representative investor combines two components: a standard mean-variance portfolio criterion (Bouchard et al., 2018) and an effect related to non-pecuniary environmental preferences. This effect is broken down into two parts. As in Pástor et al. (2021) and Zerbib (2022), it includes a preference term for companies with good environmental quality, measured by their public environmental score, E_t. However, the investor is aware of and averse to the low quality of ESG ratings (Berg et al., 2022), which can be biased by environmental communication. Therefore, she also penalizes companies for which past controversies have publicly revealed score inaccuracies using the misrating score, M_t. The investor determines her optimal asset allocation according to the following mean-variance-adjusted program:

$$\sup_{\omega \in \mathbb{A}^\omega} \mathbb{E}\left[\int_0^\infty e^{-\delta^I t} \left\{\omega_t' dS_t - \frac{\gamma}{2}\langle \omega' dS\rangle_t + \omega_t' \left(\beta E_t - \alpha M_t\right) dt\right\}\right] \tag{4}$$

where $\omega \in \mathbb{A}^\omega$ denotes the vector of quantities invested in each risky asset at time t, $\mathbb{A}^\omega$ being the set of admissible strategies for the investor, which we define formally in the proofs, $S_t \in \mathbb{R}^n$ is the asset price vector at time t, and $\gamma \in \mathbb{R}_+^*$ is the risk aversion of the investor. $\beta \in \mathbb{R}_+$ is the investor's preference sensitivity for holding green assets (also referred to as investor's *pro-environmental preferences*), $E_t \in \mathbb{R}^n$ denotes the vector of environmental scores of companies at time t, observed by the investor, $\alpha \in \mathbb{R}_+$ is the sensitivity parameter to misrating revealed by past environmental controversies, and $M_t \in \mathbb{R}^n$ is the vector of misrating proxies at time t. Finally, $\delta^I \in \mathbb{R}_+^*$ is the investor's rate of time preference. The equilibrium expected returns are determined so that the investor invests optimally and the market clears.

Program of the companies. Company i dynamically determines its optimal greening effort, v_t^i, and environmental communication effort, c_t^i, by minimizing the sum of its costs of capital, greening efforts, and communication, as follows:

$$\inf_{(v^i, c^i) \in \mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} \left(\mu_t^i + \frac{\kappa_v^i}{2}(v_t^i)^2 + \frac{\kappa_c^i}{2}(c_t^i)^2\right) dt\right], \tag{5}$$

where $\mathbb{A}$ is the set of admissible strategies for the companies, which we define formally in the proofs. The companies face quadratic greening and communication costs (Battaglini and Harstad, 2016), and κ_v^i and κ_c^i denote the marginal unit costs of greening and communication, respectively.

We specify each company's program through the minimization of its cost of capital rather than the maximization of its price for three reasons: (i) the cost of capital is a critical financial variable for companies' solvency and profitability, and it is affected by their investments in sustainable projects (e.g., Pástor et al., 2021; De Angelis et al., 2023) as well as their environmental communication (Frankovic and Kolb, 2024); (ii) consistent with McConnell and Sandberg (1975) and Nantell and Carlson (1975), the minimization of the cost of capital is a notion almost equivalent to the maximization of the initial price of the company; (iii) expected returns, which are expressed in monetary terms,[14] are homogeneous to the financial costs of environmental efforts.

3 Optimal greenwashing and investor impact

The program of the investor is solved explicitly, allowing us to derive equilibrium expected returns and allocations. For the sake of readability, all proofs are reported in the Internet Appendix 3.

Proposition 1. *The optimal asset allocation of the investor is the pointwise solution*

$$\omega_t^* = \frac{1}{\gamma}\Sigma^{-1}(\mu_t + \beta E_t - \alpha M_t),$$

and the equilibrium expected return is

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta E_t + \alpha M_t.$$

[14] As the asset prices follow a Gaussian dynamics (Equation (1)), the expected returns are *price returns* expressed in dollars.

The investor's optimal allocation breaks down into three parts: the part associated with the standard mean-variance program, $\frac{1}{\gamma}\Sigma^{-1}\mu_t$; the effect of pro-environmental preferences, $\frac{\beta}{\gamma}\Sigma^{-1}E_t$ (Pástor et al., 2021; Zerbib, 2022), which increases (decreases) the allocation in the assets with high (low) environmental scores; the new effect associated with past environmental controversies, which decreases the allocation in the assets of companies that experienced environmental controversies revealing environmental score inaccuracies, $-\frac{\alpha}{\gamma}\Sigma^{-1}M_t$.

Similarly, expected returns are also composed of the standard mean-variance component, $\gamma\Sigma\mathbf{1}_n$, adjusted for the green premium (Pástor et al., 2021; Zerbib, 2022), $-\beta E_t$, and the premium induced by misrating revealed in the past, αM_t. The greater the inaccuracies in companies' environmental scores revealed by past controversies, the higher the return investors require to hold their assets.

In view of the explicit solution for equilibrium expected returns given in Proposition 1, the optimization problem for company i takes the following form:

$$\inf_{(v^i,c^i)\in\mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\Gamma_i - \beta E_t^i + \alpha M_t^i + \frac{\kappa_v^i}{2}(v_t^i)^2 + \frac{\kappa_c^i}{2}(c_t^i)^2\right)dt\right],$$

with Γ_i a constant equal to the i-th component of the vector $\gamma\Sigma\mathbf{1}_n$.

3.1 Optimal environmental communication and greening effort

The following proposition provides a solution to this problem, which corresponds to the Stackelberg equilibrium in the game between companies and the investor, wherein the companies, choosing their greening and communication policies, play the role of the "leader," and the investor, fixing her portfolio allocation, is the "follower."

Proposition 2 (Optimal strategies)**.** *The optimal environmental communication effort, $c^{i,*}$,* *and greening effort, $v^{i,*}$, of company i are represented in feedback form as follows:*

$$c_t^{i,*} = \frac{1}{\kappa_c^i}\left(B^i - A^i(E_t^{i,*} - V_t^{i,*})\right), \tag{6a}$$

$$v_t^{i,*} = \frac{1}{\kappa_v^i}\left(\frac{\beta}{\delta} - B^i + A^i(E_t^{i,*} - V_t^{i,*})\right), \tag{6b}$$

where

$$\begin{aligned} &B^i = \frac{P^i}{Q^i}, \quad P^i = \beta(1 + \frac{A^i}{\delta\kappa_v^i}), \quad Q^i = \delta + a + b\lambda^i + \frac{2A^i}{\bar{\kappa}^i}, \quad \bar{\kappa}^i = \frac{2}{\frac{1}{\kappa_v^i} + \frac{1}{\kappa_c^i}}, \\ &A^i = \frac{\bar{\kappa}^i}{4}R^i\left(\sqrt{1 + \frac{16}{\bar{\kappa}^i}\frac{T^i}{(R^i)^2}} - 1\right), \quad R^i = \delta + 2a + \frac{2\lambda^i b}{1+b}, \quad T^i = \frac{2\lambda^i b^2\alpha}{(1+b)(\delta+\rho)}, \end{aligned} \tag{7}$$

with $E^{i,*}, V^{i,*}$ solutions of (2) when the optimal strategies $c^{i,*}, v^{i,*}$ are employed, $A^i, B^i \geq 0$ and $\frac{\beta}{\delta} - B^i \geq 0$.

Both optimal strategies are made of a positive constant part, $\frac{B^i}{\kappa_c^i}$ and $\frac{1}{\kappa_v^i}(\frac{\beta}{\delta} - B^i)$, and a stochastic part, that is linear in overrating, $E_t^{i,*} - V_t^{i,*}$, with a positive coefficient for the greening effort, $\frac{A^i}{\kappa_v^i}$, and an opposite coefficient for the communication effort, -$\frac{A^i}{\kappa_c^i}$. Moreover, note that B^i is zero if β is zero, A^i is zero if α is zero, and $\frac{\beta}{\delta} - B^i$ increases in β. These results can, therefore, be interpreted as follows. The greening and communication efforts are driven by two forces: (i) an "incentive force," which is positive and increases with the investor's pro-environmental sensitivity, β, and (ii) a "corrective force," which aims at limiting the level of misrating in response to the investor's penalty on misrating with intensity α. Overall, greening effort and environmental communication of company i jointly serve the purpose of increasing its environmental score without decoupling it too much from its fundamental environmental value.[15]

Introducing the notion of marginal benefit of a strategy allows to draw a number of

[15] To underpin this interpretation, we also provide a detailed analysis of two limiting cases in the appendix: when the investor only has pro-environmental preferences ($\beta > 0, \alpha = 0$) and when she only penalizes revealed misrating ($\beta = 0, \alpha > 0$).

additional conclusions from these results, and to deepen the understanding of the optimal greenwashing behaviour that is presented in the next subsection. We define the notion of the marginal benefit of a strategy as the impact on the opposite value of the integrated discounted cost of capital of increasing this strategy over an infinitesimal time interval. We refer to the opposite value of the integrated discounted cost of capital to define marginal benefits with respect to a maximization rather than a minimization program, which is a standard framework to interpret the notion of marginal benefits. More formally, the marginal benefit of a strategy is defined as the Fréchet derivative of the opposite value of the expected integrated discounted cost of capital in the direction of this strategy (see Definition 5 in the Internet Appendix for a precise statement). First, it is noteworthy that optimal communication and greening strategies are so that their marginal benefits equal their marginal costs, as detailed in the following proposition.

Proposition 3 (Marginal benefits of communication and greening effort)**.** *Let c^i and v^i be two admissible strategies of communication and greening effort, respectively, and let E^i be the corresponding environmental score and V^i the environmental value, solutions of equation* (2) *driven by these strategies. The marginal benefit of increasing communication at time t for company i when its environmental strategy is (c^i, v^i) is as follows:*

$$\Pi_t^{c^i,i} = \frac{\beta}{\delta + a + b\lambda^i} - 2T^i \mathbb{E}\left[\int_t^\infty e^{-(\delta+a)(s-t)} \left(E_s^i - V_s^i\right) \prod_{t \le r \le s : \Delta N_r^i \neq 0} (1 - \Theta_r^i) ds \middle| \mathcal{F}_t\right].$$

The marginal benefit of increasing green investments at time t for company i when its environmental strategy is (c^i, v^i) is as follows:

$$\Pi_t^{v^i,i} = \frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda^i} + 2T^i \mathbb{E}\left[\int_t^\infty e^{-(\delta+a)(s-t)} \left(E_s^i - V_s^i\right) \prod_{t \le r \le s : \Delta N_r^i \neq 0} (1 - \Theta_r^i) ds \middle| \mathcal{F}_t\right].$$

At optimum, communication and greening strategies, $c^{i,*}$ and $v^{i,*}$, are so that their marginal *benefits equal their marginal costs:*

$$\Pi_t^{c^{i,*},i} = \kappa_c^i c_t^{i,*}, \qquad \Pi_t^{v^{i,*},i} = \kappa_v^i v_t^{i,*}.$$

The marginal benefits of increasing communication and greening efforts at time t can be understood as follows. They both include a constant component, $\frac{\beta}{\delta+a+b\lambda^i}$ and $\frac{\beta}{\delta} - \frac{\beta}{\delta+a+b\lambda^i}$, which add up to β/δ, as well as a conditional expectation component that depends on the pair of strategies (c, v). As the sum of these conditional expectation components is zero, the sum of the two marginal benefits, $\Pi_t^{c^i,i} + \Pi_t^{v^i,i}$, is always equal to β/δ. This equality implies that the marginal benefit of the overall environmental effort of a company, including both greening and environmental communication, is the expected discounted impact on the cost of capital of increasing the environmental score, E^i: due to the investor's pro-environmental sensitivity β, the discounted integral of the cost of capital decreases by $\int_t^\infty e^{-\delta t}\beta dt = \beta/\delta$.

In particular, at the optimum, the marginal cost of the overall environmental effort, $\kappa_c^i c_t^{i,*}$ $+ \kappa_v^i v_t^{i,*}$, is equal to β/δ:

$$\kappa_v^i v_t^{i,*} + \kappa_c^i c_t^{i,*} = \frac{\beta}{\delta}. \tag{8}$$

Hence, the investor's penalty on revealed misrating does not influence the marginal spending in total environmental efforts but only determines the distribution of efforts between greening and environmental communication.

Interestingly, the combination of the investor's pro-environmental preferences, $\beta > 0$, and misrating penalty, $\alpha > 0$, induces a complementarity in the use of the two environmental strategies of the company. This can be understood through the following reasoning, developed in two steps: first, we interpret our results when the investor only has pro-environmental preferences or only penalizes misrating, and second, we focus on the general case where she both has pro-environmental preferences and penalizes misrating.

First, as stated in the next proposition, both strategies are perfect substitutes when the investor only has pro-environmental preferences, or only penalizes misrating. This proposition relies on the notion of marginal rate of substitution between strategies, which is defined by the ratio of their marginal benefits.

Definition 2 (Marginal rate of substitution)**.** The marginal rate of substitution between greening effort and environmental communication for company i at time t, $\mathrm{MRS}_t^{v \to c,i}$, equals the ratio of their marginal benefits as follows:

$$\mathrm{MRS}_t^{v \to c,i} = \frac{\Pi_t^{v^i,i}}{\Pi_t^{c^i,i}}.$$

The marginal rate of substitution from greening to communication describes how many additional communication effort should be done to replace one unit of greening effort while keeping the same benefit. When the marginal rate of substitution between two strategies does not depend on the quantity of efforts that has already been made (i.e., does not depend on c or v), the two strategies are called perfect substitutes: one strategy can be infinitely replaced by the other one if this is justified by pure costs or benefits considerations. The next proposition shows that this is the case when the investor only applies one of her two preference features.

Proposition 4 (Marginal rate of substitution)**.** *When the investor only has pro-environmental preferences ($\beta > 0, \alpha = 0$), the marginal rate of substitution between greening effort and environmental communication is constant, as follows:*

$$MRS_t^{v \to c,i} = \frac{a + b\lambda^i}{\delta}.$$

When there is the investor penalty only ($\beta = 0, \alpha > 0$), it is also constant, as follows:

$$MRS_t^{v \to c,i} = -1.$$

As these marginal rates of substitution are both constant, the two environmental strategies are perfect substitutes in these two cases.

When the investor only has pro-environmental preferences ($\beta > 0, \alpha = 0$), a company can decrease its cost of capital by increasing its environmental score. To do so, the company can either raise its communication or its greening effort. More precisely, the company must replace one unit of greening effort by $\frac{a+b\lambda^i}{\delta}$ units of environmental communication to keep the same benefit on its expected discounted cost of capital. As this ratio does not depend on the two strategies, c^i_t and v^i_t, both strategies are perfect substitutes: the company chooses indifferently between one or the other strategy, depending on their relative costs and abilities to raise the environmental score.

The average rate at which the greening effort impacts the environmental score depends on the quality of information about the fundamental environmental value of the company, which can be characterized by the following notion of "revelation intensity."

Definition 3 (Revelation intensity)**.** We refer to $a + b\lambda^i$ as the "revelation intensity" of the environmental score.

The revelation intensity combines the effort of the rating agency, which pushes the environmental rating towards the fundamental environmental value with speed a, with the discontinuous effect of controversies, which act as revealing events, where a portion b of misrating is revealed with intensity λ^i. This quantity represents the average rate at which the fundamental environmental value, V^i, translates into the environmental score, E^i (Equation (2a)), which is also the rate at which the influence of misleading green communication vanishes from the environmental score, E^i. In the rest of the paper, we assume that $a + b\lambda^i$ is strictly positive, which means that at least a minimum amount of information about the environmental value is revealed, on average, at each point in time and for each company.

When the investor only penalizes misrating ($\beta = 0, \alpha > 0$), companies only care about reducing their misrating to decrease their cost of capital. In this context, both strategies, while playing in opposite directions, have the same efficiency at getting the environmental score and the environmental value close to each other: environmental communication can be used to drive the environmental score in one direction, and greening efforts to drive the company's environmental value in the other, with the same impact on the spread between the environmental score and the environmental value. As the marginal rate of substitution between the two strategies is constant, both strategies are, again, perfect substitutes. The relative use of these two strategies by a company depends only on their relative marginal costs.

In the second step of our interpretation, we highlight that the combination of the investor's pro-environmental preferences and the misrating penalty induces a complementarity between the two environmental strategies (see Figure 1 based on the calibration detailed in the Internet Appendix 4): the average environmental communication now decreases with the marginal unit cost of greening effort, as illustrated in Figure 1. This complementarity can be understood by the combined incentive of both pro-environmental preferences and misrating penalty: a company wants to increase its environmental score without moving it too far away from its fundamental environmental value. Hence, when the cost of greening effort increases, it cannot be purely substituted by communication, as this would induce a high misrating. Instead, the company must keep its communication effort not too far from its greening effort, which translates into a reduction in both types of efforts.

3.2 Optimal greenwashing strategy

In this subsection, we characterize the condition under which companies greenwash, the optimal effort of greenwashing, and the impact of greenwashing on the environmental score.

Proposition 5 (Greenwashing effort)**.** *When the following condition is satisfied,*

$$\frac{\kappa_v^i}{\kappa_c^i} > \frac{a + b\lambda^i}{\delta} \tag{9}$$

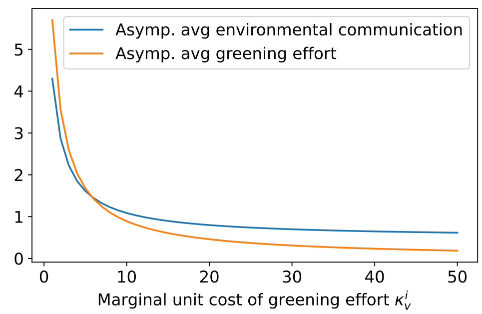


Figure 1: **Average environmental communication and greening as a function of κ_v^i.** This figure illustrates the asymptote of the expected optimal environmental communication, $\lim_{t\to\infty}\mathbb{E}[c_t^{i,*}]$, and greening effort, $\lim_{t\to\infty}\mathbb{E}[v_t^{i,*}]$, as a function of the marginal unit cost of greening effort, κ_v^i. The calibration is given in the Internet Appendix 4.

company i greenwashes as long as its overrating, $E_t^{i,*} - V_t^{i,*}$, is not too high: specifically, it greenwashes if, and only if, $0 \leq E_t^{i,*} - V_t^{i,*} < \frac{1}{\frac{2}{\bar{\kappa}^i}A^i}G_{max}^i$, where $G_{max}^i = \frac{2}{\bar{\kappa}^i}B^i - \frac{\beta}{\delta\kappa_v^i}$. Its greenwashing effort, $c_t^{i,*} - v_t^{i,*}$, is maximal, equal to the positive quantity G_{max}^i, when its score is equal to the fundamental environmental value, $E_t^{i,*} = V_t^{i,*}$, and decreases linearly in the overrating, $E_t^{i,*} - V_t^{i,*}$, with slope $-\frac{2}{\bar{\kappa}^i}A^i$, reaching 0 when the overrating equals $\frac{1}{\frac{2}{\bar{\kappa}^i}A^i}G_{max}^i$.

When condition (9) is not satisfied, company i never greenwashes.

We interpret this proposition in several steps. First, when the representative investor both has pro-environmental preferences and penalizes revealed environmental misrating, company i greenwashes under the "ON-OFF" condition (9). This condition compares the ratio of marginal benefits of the two strategies, $(a + b\lambda^i)/\delta$, with their relative marginal unit costs, κ_v^i/κ_c^i: when it is sufficiently cheap to engage in environmental communication relative to greening efforts, when the quality of information about the fundamental environmental value is sufficiently low, or when the company's rate of time preference is high enough, the company greenwashes. Hence, the decision to greenwash does not depend on the investor's penalty, but solely on a condition guaranteeing that it is more beneficial to communicate than to abate to raise the environmental score, even when the company is overrated and misrating is penalized. However, the amount of greenwashing effort depends on the investor's misrating penalty.

The greenwashing effort decreases linearly with the company's overrating, $E_t^{i,*} - V_t^{i,*}$, through the parameter A^i, which represents the "corrective force" due to the penalty. Therefore, the occurrence of an environmental controversy revealing a portion of the company's overrating triggers both a drop in its environmental score and an increase in its greenwashing effort. This effect echoes the empirical findings of Duchin et al. (2023), providing evidence for greenwashing following an "environmental risk incident." A related consequence is that company i greenwashes the most when its environmental score correctly reflects the environmental value, that is, when $E_t^{i,*} = V_t^{i,*}$.

Company i no longer greenwashes once the level of overrating, $E_t^{i,*} - V_t^{i,*}$, exceeds the greenwashing threshold, $\frac{1}{\frac{2}{\bar{\kappa}^i} A^i} G^i_{max}$. When company i's overrating is above this threshold, the company allows its overrating to decrease (i) through the action of the rating agency and (ii) by communicating less than abating. Hence, company i's greenwashing effort is dedicated to maintaining its overrating at a certain positive target. This can be further understood by assessing the impact of this greenwashing strategy. To do so, we start by introducing the notion of greenwashing impact.

Definition 4 (Greenwashing impact)**.** The impact of greenwashing is the asymptotic value of the expected spread between the environmental score, E^i, and the environmental value, V^i. Formally, it writes as follows:

$$\lim_{t \to \infty} \mathbb{E}[E_t^i - V_t^i].$$

The expectation is taken to average out the measurement error. In addition, we consider the asymptotic value because this expected spread tends very quickly towards a limit value with a simple and informative expression, as illustrated in the next proposition. As a result of this greenwashing strategy, the overrating of company i quickly converges, on average, towards a positive quantity that is related to its greenwashing threshold.

Proposition 6 (Greenwashing impact)**.** *When condition* (9) *is satisfied, the impact of company i's greenwashing strategy can be measured as:*

$$\lim_{t\to\infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = \frac{1}{\frac{2}{\bar{\kappa}^i} A^i + a + b\lambda^i} G_{max}^i,$$

where the convergence takes place with an exponential rate.

Consistent with our interpretation of the optimal greenwashing effort, the optimal greenwashing strategy induces a positive bias on the environmental score of company i.

3.3 Investor impact

In this subsection, we use our characterization of greenwashing to identify how investors can curb greenwashing practices. As shown in the proposition below, the investor can have an impact on corporate greenwashing.

Proposition 7 (Investor's impact on greenwashing)**.** *When condition* (9) *is satisfied, the maximal greenwashing effort,* G_{max}^i*, is linearly increasing in* β *and decreasing and convex in* α*.*

The effort and impact of greenwashing both increase in the pro-environmental preferences of the investor through G_{max}^i, as these preferences spur companies to display a higher environmental score. However, the investor has the ability to curb greenwashing effort and impact: by increasing her sensitivity to misrating, α, the investor lowers companies' maximal greenwashing efforts, their greenwashing thresholds, and the impact of their greenwashing strategies, which all depend on $(G_{max}^i, 1 \leq i \leq n)$. This translates into a lower average greenwashing effort, as illustrated in Figure 2a.[16]

[16] As the greenwashing and greening efforts are linear deterministic functions of the overrating, $E_t^{i,*} - V_t^{i,*}$, their expectations also converge at an exponential rate toward their asymptotic values (see Proposition 6). This justifies the use of these asymptotic values to analyse average greenwashing and greening efforts.

In the proposition below, we show how the misrating penalty also affects the optimal greening effort of companies.

Proposition 8 (Investor's impact on greening effort)**.** *The constant part of the optimal greening effort,* $\frac{1}{\kappa_v^i}\left(\frac{\beta}{\delta} - B^i\right)$*, increases linearly in* β*, and, when condition* (9) *is satisfied, is concave and increasing in* α*.*

This proposition highlights the positive impact of investors' penalties for environmental misrating on companies' greening strategies. In addition to the investor's pro-environmental preferences, which promote greening efforts, penalizing environmental misrating not only reduces greenwashing but also further increases greening efforts. In particular, even a small misrating penalty appears to have a significant effect on the greening effort (Figure 2b). This result adds to the emerging literature on impact investing (Landier and Lovo, 2023; Green and Roth, 2024; Pástor et al., 2022; De Angelis et al., 2023; Oehmke and Opp, 2024) by identifying an effective vector available to investors to encourage companies to mitigate their environmental footprints.

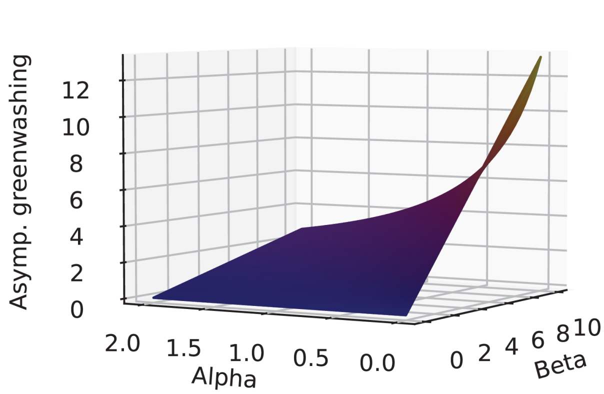
3.4 Complementary tools to curb greenwashing

Our model allows us to identify policy tools that could, as a complement to investor action, contribute to curbing greenwashing, namely (i) increasing transparency and (ii) fostering technological innovation in green technologies. As it is not possible to carry out an analytical analysis of these tools, their effects are illustrated through numerical sensitivity analyses.

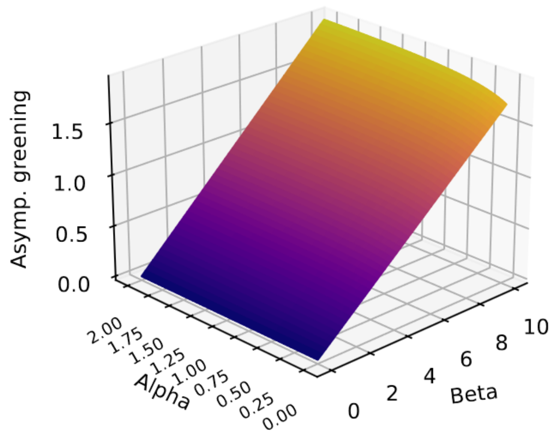
3.4.1 Regulations increasing transparency

In this subsection, we investigate to what extent policies playing on the transparency parameters can be alternative or complementary tools to the penalization of misrating by investors.

When the investor does not penalize the observed misrating, increasing the revelation intensity can strongly deter companies from engaging in greenwashing.



(a)



(b)

Figure 2: **Average greenwashing and greening as a function of α and β.** This figure displays the asymptotic expected optimal greenwashing ($\lim_{t\to\infty} \mathbb{E}[c_t^* - v_t^*]$; figure a) and greening ($\lim_{t\to\infty} \mathbb{E}[v_t^*]$; figure b) efforts as a function of the pro-environmental sensitivity, β, and the misrating penalty, α. The calibration is given in the Internet Appendix 4.

Figure 3 illustrates the effect of moving each of the transparency parameters (a, b, λ^i) separately on greenwashing efforts and impacts, using the baseline calibration (Internet Appendix 4). Increasing the power of the rating agency in recovering the true environmental information through parameter a decreases substantially the greenwashing effort. In addition, increasing any transparency parameter amplifies the mitigation of the greenwashing impact; indeed, a higher revelation intensity does not only deter greenwashing practices, but also makes its effect on the environmental score less durable.

However, when the investor sufficiently penalizes the observed misrating, the action of the rating agency is not an efficient complementary tool: increasing a does not significantly reduce greenwashing efforts and impacts (Figures 4a and 4d). Conversely, the revelation of controversies is a strong complementary tool to the investor penalty of misrating: a minimum level of intensity (λ^i) and amplitude (b) of revelation is necessary to channel the effect of the investor penalty; the mitigating effect of the penalty on greenwashing efforts and impacts significantly increases with λ^i and b (Figures 4b, 4c, 4e, 4f). Therefore, increasing

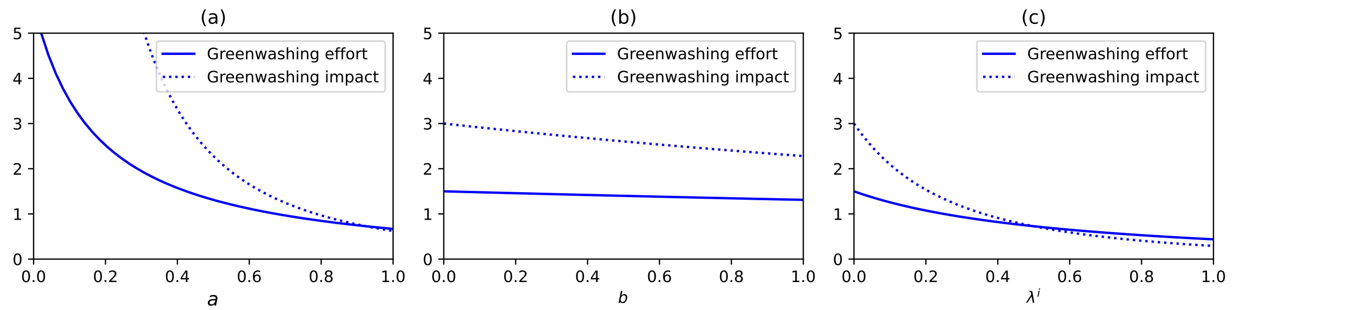


Figure 3: **Greenwashing effort and impact and transparency parameters when $\alpha = 0$.** This figure displays the greenwashing effort, G_i^β, (solid lines), and greenwashing impact, $\lim_{t\to\infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}]$, (dotted lines), as a function of transparency parameters a, b, λ^i, when the investor's penalty, α, is null. The reference calibration is given in the Internet Appendix 4.

the investigating power of stakeholders (hence, contributing to increasing λ^i), or triggering an in-depth re-assessment of a company's environmental footprint once it is suspected to be overrated (hence, contributing to increasing b) would complement and increase the impact of investor action by raising the pressure on companies to reduce their greenwashing practices.

3.4.2 Green technological change

Can green technological change help curb greenwashing? Figure 5 suggests that the marginal unit cost of greening needs to decrease substantially before its impact on greenwashing practices becomes significant. Indeed, companies no longer practice greenwashing when the relative marginal unit cost of greening versus communication is sufficiently low, that is when the inequality (9) is no longer satisfied ("ON-OFF" greenwashing condition): in the central calibration, this happens when this ratio drops to 5.7. This result shows that maintaining a sustained and pronounced research and development effort to bring down the marginal costs of new green technologies (Popp, Santen, Fisher-Vanden, and Webster, 2013) would, in addition to increasing greening efforts (Figure 1), simultaneously help curb corporate greenwashing practices.

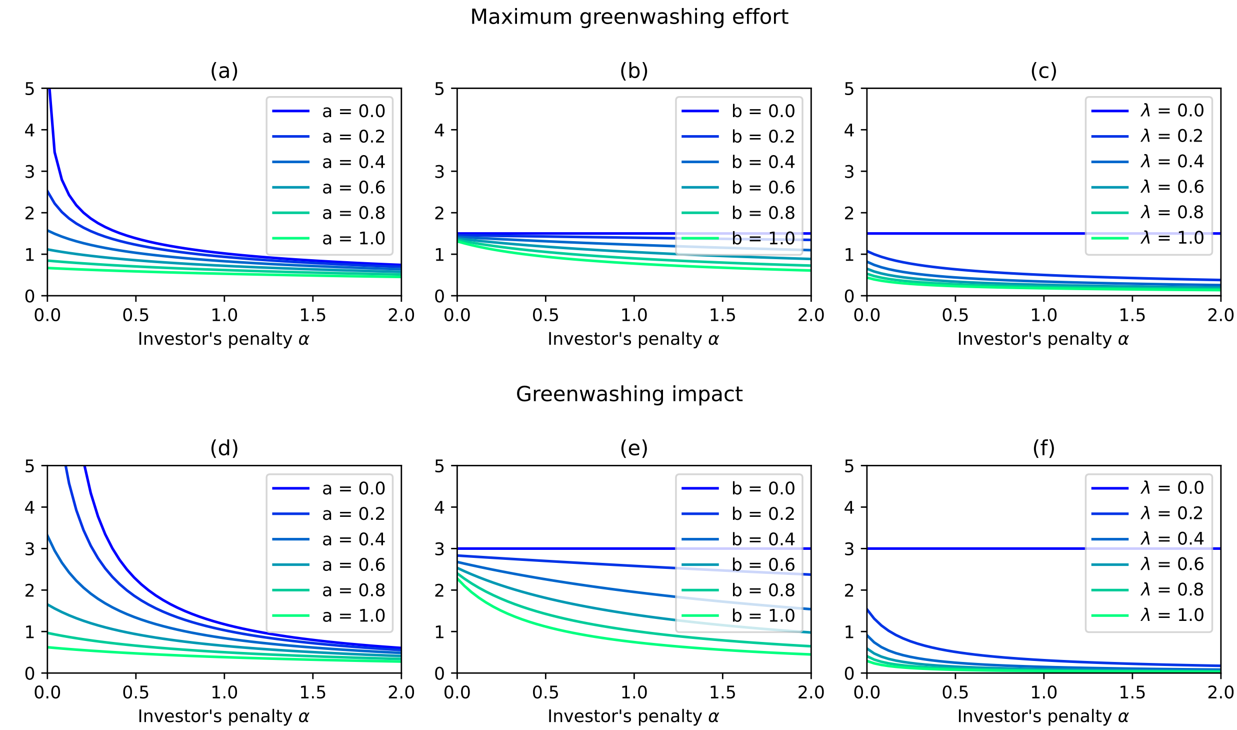


Figure 4: **Greenwashing effort and impact, penalty α and transparency parameters.** This figure displays the maximum greenwashing effort, G^i_{max}, (first row), and greenwashing impact, $\lim_{t\to\infty}\mathbb{E}[E^{i,*}_t - V^{i,*}_t]$, (second row), as a function of the investor's penalty, α, for different values of transparency parameters a, b, λ^i. The reference calibration is given in the Internet Appendix 4.

4 Introducing interaction between companies

Instead of caring about the absolute environmental value of each company, the investor may prefer to tilt her portfolio towards the greenest companies in the investment universe. In this section, we present an extension of the investor's program presented in Section 2, in which the environmental score of each company is scaled by the average environmental score of all companies. Through an adjustment of equilibrium expected returns, this change introduces an interaction between the firms' objectives leading to an n-player game.

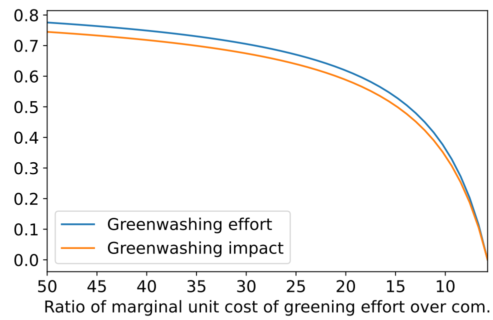


Figure 5: **Greenwashing and technological change.** Maximum greenwashing effort, G^i_{max}, and impact, $\lim_{t\to\infty}\mathbb{E}[E^{i,*}_t - V^{i,*}_t]$, in function of the ratio of marginal unit costs of greening and communication κ^i_v/κ^i_c. Consistently with Proposition 5, greenwashing is zero when the threshold represented by equation (9) is hit.

The n-player game The investor's extended program is set as follows:

$$\sup_{\omega\in\mathbb{A}^\omega}\mathbb{E}\left[\int_0^\infty e^{-rt}\Big\{\omega'_t dS_t - \frac{\gamma}{2}\langle\omega' dS\rangle_t + \omega'_t\big(\beta\frac{E_t}{h(\frac{1}{n}\sum_j E^j_t)} - \alpha M_t\big)dt\Big\}\right],$$

with h a regular function bounded from below by a strictly positive constant and approximating the identity function on $\mathbb{R}_+$.[17] This new specification is realistic as (i) rating agencies regularly rescale the environmental scores[18] and (ii) ESG investors often follow a "best-in-class" investment strategy, usually at the sector level. Notice that, when h is a constant function equal to one, this program boils down to the one in Section 2.

Similarly to the initial problem, equilibrium expected returns are easily deduced from

[17]This regularization function is applied as, due to the specified dynamics of the environmental score (equation (2a)), this score could potentially take negative or zero values. However, in practice, the probability that such irrealistic values are taken can be made negligible through an appropriate calibration.

[18]For example, MSCI ESG ratings are industry-adjusted according to an industry benchmark, which is revised at least once a year (https://www.msci.com/documents/1296102/34424357/MSCI+ESG+Ratings+Methodology.pdf). Moreover, Refinitiv LSEG ESG scores are the direct result of a cross-sectional comparison between companies' raw metrics, which are ranked to calculate percentile scores (https://www.lseg.com/content/dam/data-analytics/en_us/documents/methodology/lseg-esg-scores-methodology.pdf).

this new program. They are expressed as follows:[19]

$$\mu_t = \gamma\Sigma\mathbf{1}_n - \beta\frac{E_t}{h(\frac{1}{n}\sum_j E_t^j)} + \alpha M_t. \tag{10}$$

Plugging these new equilibrium expected returns in company i's program gives the following:

$$\inf_{(v^i,c^i)\in\mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\Gamma_i - \beta\frac{E_t^i}{h(\frac{1}{n}\sum_j E_t^j)} + \alpha M_t^i + \frac{\kappa_v}{2}(v_t^i)^2 + \frac{\kappa_c}{2}(c_t^i)^2\right)dt\right].$$

Companies' programs are now interacting through the average environmental score of all companies. Moreover, they are no longer linear quadratic: each company controls both the numerator and the denominator in the term involving its environmental score, $E_t^i/h(\frac{1}{n}\sum_j E_t^j)$. Therefore, to approximate the Nash equilibrium of this n-player game with interpretable quantities, we formulate and solve the mean field limit of this game, in other words, the limit obtained by making the number of companies n tend to infinity. At the mean field limit, a generic company does not have any impact on the average environmental score in the investment universe and its optimization problem becomes a linear quadratic program, in which the average environmental score is a time-dependent deterministic parameter.

To be able to set up a mean field game (MFG) that approximates the greenwashing n-player game, we need to make two additional assumptions. (i) Companies are homogeneous: all parameters are the same for each company. (ii) Their environmental scores are driven by idiosyncratic noises: $(W^i, N^i)_i$ are assumed to be independent and identically distributed. Under these additional assumptions, in the Internet Appendix, we prove that there exists a unique Nash equilibrium at the mean field limit when the problem has a finite horizon.

More specifically, in the Internet Appendix, we derive the equilibrium expected returns (10); under the additional assumptions (i) and (ii), we define and then demonstrate the existence and uniqueness of the mean field equilibrium (MFE; mean field version of a Nash

[19]The main technical results and all the proofs of this section can be found in the Internet Appendix.

equilibrium) in the limit Greenwashing mean field game with finite horizon; finally, we discuss the algorithm approximating the unique MFE of the Greenwashing MFG and show how its convergence can be controlled.

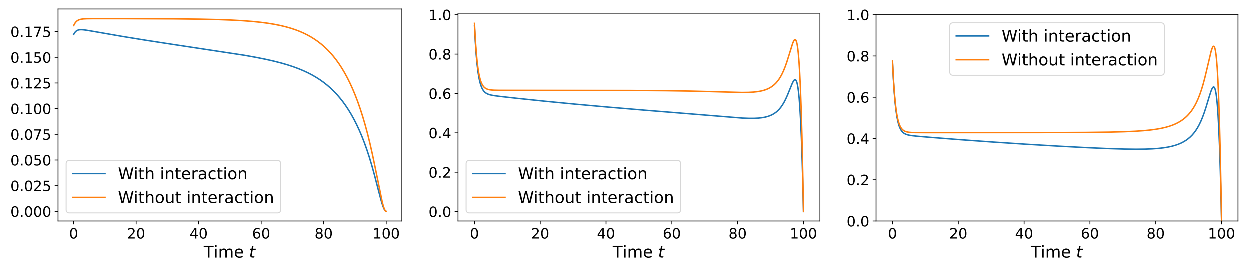
Results We derive two types of results. First, analytically, we express the optimal strategy of communication and greening of a generic company. We find that, at the mean field equilibrium, the optimal environmental strategy follows a similar pattern as that in the baseline case (see the Internet Appendix, Proposition 15). In particular, optimal efforts follow the structure of those in Proposition 2 with similar linear coefficients B, A playing the same roles, with the same signs, but being now time-dependent: under the baseline calibration (Internet Appendix 4),[20] the normalization of companies' environmental scores leads to positive greening, communication, and greenwashing efforts, as in the baseline case without normalization (Figure 6).[21] These results confirm the robustness of our main qualitative conclusions.

Second, however, the numerical analysis shows that all these efforts are lower and lead to a smaller increase in the environmental scores of companies over time compared to the baseline case (Figure 7). Indeed, as the investor only values relative scores, companies have less incentive to push their environmental scores as high as possible: the decrease in their costs of capital only stems from a comparison of their environmental scores to those of their peers. These results suggest that the cross-sectional normalization of firms' environmental scores by rating agencies and the best-in-class approaches to portfolio selection have a mixed effect: they have a detrimental impact on the environmental efforts of companies but contribute to

[20]We add to the baseline calibration described in the Internet Appendix 4 the initial value of the environmental score and the environmental value of the company, both set to 50. In addition, to allow comparability with the case without interaction, we modify one parameter in the baseline calibration described in the Internet Appendix 4: β is changed to 50, so that companies have the same incentive to increase their environmental score at the initial date, whether or not their scores are normalized. Finally, time horizon is set to 100, as it is enough to reach some stationary pattern between the initial and terminal conditions.

[21]Due to the finite time horizon, the model is not stationary anymore. Thus, peaks arise at the beginning and the end of the time period on each graph, which are due to the impact of the initial and terminal conditions, and are not of interest in the present study.

mitigating greenwashing.



(a) (b) (c)

Figure 6: Average greening (a), communication (b) and greenwashing (c) efforts with and without interaction (*blue* and *orange* curves respectively).

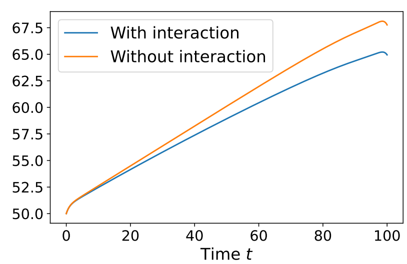


Figure 7: Average environmental score with and without interaction between the companies.

5 Empirical evidence

In this section, we carry out an empirical analysis of companies' environmental communication flow, c, at a global level to support the results of our model. We focus our empirical study on environmental communication and not on greenwashing directly, as we do not have access to sufficiently robust data on the dynamics of the greening policies of all the companies studied. We document two main results. First, average environmental communication is positive in almost all the months studied. Second, the empirical dynamics of environmental communication are consistent with those highlighted in the model. We conclude this section by suggesting interpretations regarding corporate greenwashing practices.

5.1 Data

We use data from Covalence, a data provider that constructs an environmental performance score, an environmental reputation score, and an environmental controversy score, based on data from published news, for companies worldwide, at a monthly frequency.[22] We denote these scores by E_t^i, Rep_t^i and Con_t^i, respectively, for company $i \in \{1, \ldots, n\}$, available at the end of month $t \in \{1, \ldots, T\}$. All of them are between 0 and 100. Consistent with the rise in environmental awareness among investors after the Paris Agreement, our main analysis covers a scope of 3,769 global companies covered by Covalence between December 2015 and December 2022, representing 145,508 firm×month observations. The description of and statistics on all the variables used in the empirical analysis are available in Internet Appendix 6 (Table 6.11).

A first analysis of the correlation between monthly variations in environmental reputation and environmental score reveals an intriguing dynamic: the proportion of companies showing a negative correlation between variations in their environmental reputation and their previous month's environmental score varies between 63% and 78% over the years (Figure 8). This correlation echoes the countercyclical dynamics of communication in relation to the environmental score highlighted in the model. To explore this point in greater detail, we develop the empirical analysis below.

5.2 Identification strategy

We develop a two-stage estimation method to analyze the cross-section of companies' environmental communications and provide support for the results of our model.

[22]Covalence is a Switzerland-based data provider, founded in 2001, which produces ESG reputation data using media monitoring, artificial intelligence, and human analysis (`https://www.covalence.ch/`). Its services are used by asset managers, asset owners, international organizations and institutions (e.g., the EU, the WWF), and academic institutions. Its datasets have been used by several influential papers (e.g., Daubanes and Rochet, 2019). The construction methods of the indices are available in the White Paper available at this URL: `https://www.covalence.ch/docs/Covalence_GreenwashingRiskIndicator_WhitePaper.pdf`.

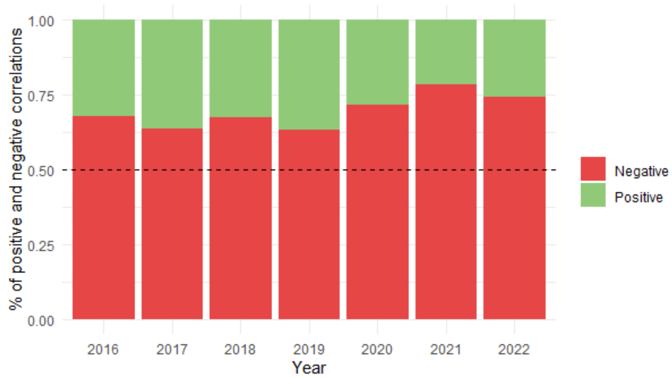


Figure 8: **Correlation between changes in environmental reputation and environmental score.** This figure depicts the average correlation between the monthly changes in environmental reputation and the previous month's environmental score between 2016 and 2022 for all companies in the universe.

First step. In the first step, we build a proxy for the monthly environmental communication flow. Building such a proxy involves two challenges. First, as the environmental reputation score is driven by both companies' environmental communication and the controversies that affect them,[23] we construct an environmental communication *score* purged of the effect of environmental controversies, which is defined as the orthogonal component of the environmental reputation score to the environmental controversy score through a Within regression, that is, for company i at the end of month t, $\alpha_1^i + \varepsilon_{1,t}^i$ in Equation (11) below. Second, given the simultaneity of the reputation and controversy scores, we instrument company i's environmental controversy score at the end of month t by company i's environmental controversy score at the end of month $t-1$. More precisely, we estimate the following specification:

$$Rep_t^i = \alpha_1^i + \beta_1 Con_t^{i,*} + \varepsilon_{1,t}^i, \tag{11}$$

where $Con_t^{i,*}$ is the prediction of the following regression: $Con_t^i = \alpha_2^i + \beta_2 Con_{t-1}^i + \varepsilon_{2,t}^i$.

[23]More details on the construction method of the forward-looking reputation indicator are available on page 4 of the White Paper by Covalence: https://www.covalence.ch/docs/Covalence_GreenwashingRiskIndicator_WhitePaper.pdf

The instrument Con_t^i verifies the relevance condition: the R^2 of the regression of Con_t^i on Con_{t-1}^i is 76.4%, and the correlation between both variables is 81.3%. In addition, the weak exogeneity condition is satisfied. Indeed, the shocks to environmental reputation scores at the end of month t, $\varepsilon_{1,t}^i$, are uncorrelated with controversies that took place during month $t-j$, with $j \in \{1, \ldots, t-1\}$.[24]

The Within estimation under weak exogeneity carries a bias that tends to zero as the number of periods increases, as shown in Lemma 3.9 (Internet Appendix 3.5). Here, we perform the estimation on $T = 84$ months in the baseline case and $T = 120$ months in a robustness test.

By construction, the environmental communication score in month t embeds information on environmental communication from the past months. Since c_t^i is the flow of firm i's environmental communication during month t, we approximate it as the difference in the environmental communication score between the end of month t and the end of month $t-1$:

$$\hat{c}_t^i \equiv \left(\hat{\alpha}_1^i + \hat{\varepsilon}_{1,t}^i\right) - \left(\hat{\alpha}_1^i + \hat{\varepsilon}_{1,t-1}^i\right) = \hat{\varepsilon}_{1,t}^i - \hat{\varepsilon}_{1,t-1}^i \tag{12}$$

Second step. In the second step, we provide empirical support for the environmental communication dynamics highlighted by the model, focusing on the time derivative of Equation (6a). Indeed, the fundamental environmental values of companies are unknown and probably correlated with companies' environmental scores. However, it is reasonable to assume that these values are highly inert from one month to the next. Thus, we set

$$\frac{1}{\kappa_c} A^i \Delta V_t^i = \eta_1^i + \eta_{2,t}^i, \tag{13}$$

[24]Formally, $\forall i \in \{1, \ldots, n\}, \forall (t', t) \in \{1, \ldots, T\}^2, t' \geq t, \mathbb{E}(\varepsilon_{1,t'}^i Con_t^{i,*}) = 0$, because $\forall i \in \{1, \ldots, n\}, \forall t \in \{1, \ldots, T\}, \forall j \in \{1, \ldots, t-1\}, \mathbb{E}(\varepsilon_{1,t}^i Con_{t-j}^i) = 0$.

with η_1^i a constant likely to be close to zero and $\eta_{2,t}^i$ an error term, and we focus on the following general specification based on the first differences of the variables:

$$\Delta \hat{c}_t^i = \alpha_3^i + \tau_{3,t} + \beta_3 \Delta E_t^i + \varepsilon_{3,t}^i, \tag{14}$$

where $\Delta \hat{c}_t^i$ is the change in communication flow between month t and month $t+1$, and ΔE_t^i is the change in environmental score between month t (set at the end of month $t-1$) and month $t+1$ (set at the end of month t). To the first difference of the equilibrium equation, we add time fixed effects, $\tau_{3,t}$, to control for unobserved time heterogeneity.

Given the simultaneity between the change in communication flow, $\Delta \hat{c}_t^i$, and the change in environmental score, ΔE_t^i, as the communication flow at date $t-1$ could influence the environmental score at date t, we instrument the change in environmental score with the environmental score available throughout month $t-1$ and calculated at the end of month $t-2$, E_{t-2}^i. Therefore, we estimate a Within regression with robust standard errors based on the following specification:

$$\Delta \hat{c}_t^i = \alpha_3^i + \tau_{3,t} + \beta_3 \Delta E_t^{i,*} + \varepsilon_{3,t}^i, \tag{15}$$

where $\Delta E_t^{i,*}$ is the prediction of the following regression: $\Delta E_t^i = \alpha_4^i + \tau_{4,t} + \beta_4 E_{t-2}^i + \varepsilon_{4,t}^i$.

So as to draw robust conclusions from the empirical analysis, we carry out the estimations on several samples: the entire universe of companies, as well as the $10\%, 20\%, ..., 90\%$ of companies with the lowest environmental score within each sector for each month, and the $10\%, 20\%, ..., 90\%$ of companies with the highest environmental score within each sector for each month. For all these samples, the instrument is relevant and strong (see Tables 6.3 and 6.4 in the Internet Appendix 6). In addition, the weak exogeneity condition is satisfied. Indeed, we can reasonably assume that the shocks to the change in communication flow between month t and month $t+1$, $\varepsilon_{3,t}^i$, are uncorrelated with the environmental scores set at the end of month $t-j$, with $j \in \{2, \ldots, t-1\}$.[25] As the estimation is performed at a monthly frequency over 84 months, with a robustness test over 120 months, the bias of the Within estimate under weak exogeneity is likely to be low (Lemma 3.9).

We perform a battery of complementary estimations, including the addition of monthly systematic risk and return controls, $\beta_{t-1}^{CAPM,i}$ and R_{t-1}^{i}, respectively, estimated at the end of month $t-1$ and available throughout month t, in Specification (15). As a proxy for systematic risk, we use a 12-month rolling CAPM beta, $\beta_t^{CAPM,i} = Var^{-1}(R_t^m)Cov(R_t^i, R_t^m)$, where R^i and R^m denote firm i's return and the market return, respectively. We also repeat the estimation by starting the analysis period at different dates as well as performing the estimation on several environmental sub-scores.

5.3 Estimations

The regression of the environmental reputation score on the instrumented environmental controversy score (first step, Specification (11)) yields a highly significant $\hat{\beta}_1 = 0.04$ (Table 6.2, Internet Appendix 6). This estimation allows us to retrieve the fixed effects, $\hat{\alpha}_1^i$, and the residuals, $\hat{\varepsilon}_{1,t}^i$, the sum of which is a proxy for the environmental communication score.

The empirical analysis allows us to document two main results. First, the proxy for the monthly environmental communication flow, c, shows that 98.8% of the average environmental communication over the period is positive. On average, companies engage almost structurally in *green communication* as defined in the theoretical section (Figure 9).

Second, we find significant empirical evidence supporting the dynamics of environmental communication derived from the theoretical section. We carry out the second-step estimation, whose results are presented in Tables 1. As expected, $\hat{\beta}_3$ is negative and highly significant: in the entire sample, the beta is -0.119 and the t-stat is -3.6. In addition, as presented in Tables 6.5 and 6.6 in the Internet Appendix, the significantly negative estimate

[25]Formally, $\forall i \in \{1, \ldots, n\}, \forall (t', t) \in \{1, \ldots, T\}^2, t' \geq t$, $\mathbb{E}(\varepsilon_{3,t'}^i \Delta E_t^{i,*}) = 0$, because $\forall i \in \{1, \ldots, n\}, \forall t \in \{1, \ldots, T\}, \forall j \in \{2, \ldots, t-1\}$, $\mathbb{E}(\varepsilon_{3,t}^i E_{t-j}^i) = 0$.

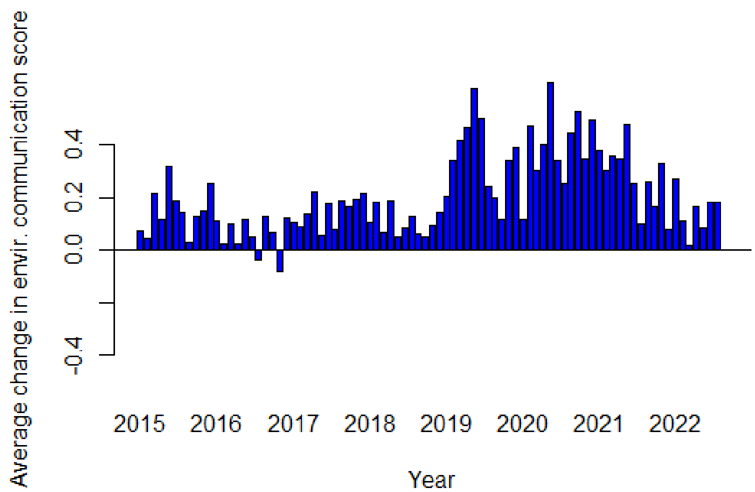


Figure 9: **Environmental communication.** This figure shows the average monthly flow of environmental communication, $\bar{\bar{c}}$.

is robust to focusing on several subsamples: the $10\%, 20\%, ...,$ and 90% brownest companies in the universe have a beta ranging from -0.07 to -0.24, with t-statistics below -2.5 for all samples except the top 10% brownest companies, even reaching -4.5 for the 50% brownest companies. Moreover, the $10\%, 20\%, ...,$ and 90% greenest companies in the universe have a beta ranging from -0.24 to -0.45 and t-statistics below -3, even reaching -7.7 for the 40% greenest companies. Therefore, the empirical findings suggest that companies, especially the greenest ones,[26] use environmental communication in a counter-cyclical way with respect to the evolution of their environmental score, in line with the results of the model.

The results are also robust to the introduction of controls for systematic risks and returns (Table 6.7 and 6.8)[27]. We carry out other robustness tests by repeating the estimation starting at different dates: December 2012, December 2017, December 2019, and December 2021. The estimate $\hat{\beta}_3$ remains strongly significant (see Tables 6.9). Finally, we repeat the

[26] This difference may be related to the fact that the most polluting companies are subject to greater media and stakeholder attention than the greenest ones. This hypothesis opens up an interesting avenue for future research.

[27] Except for the 10% and 20% brownest companies as well as the 10% greenest companies, the estimations on all subsamples and the entire sample yield significant and consistent estimates.

Table 1: **Main estimation.** This Table gives the result of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$. The standard deviation is shown in brackets below the estimate.

	Dependent variable: $\Delta\hat{c}_t^i$
$\Delta E_t^{i,*}$	-0.119^{***}
	(0.033)
Firm FE	Yes
Month FE	Yes
Observations	145,508
R^2	0.017
Adjusted R^2	-0.008
F Statistic	0.661

Note: *p<0.1; **p<0.05; ***p<0.01

estimation applied to the three environmental subscores calculated by Covalence, which are related to (i) the environmental impacts of the products sold, (ii) the resources used, and (iii) the emissions, effluents, and waste. For all three subscores, the results are robust (see Table 6.10 showing the example of the 50% brownest and 50% greenest companies of the universe).

The two main results in this empirical analysis support the two main forces at play in the environmental communication dynamics: the "incentive force" is supported by the finding of a quasi-structural green communication policy, while the "corrective force" is supported by the finding of a countercyclical dynamic of the environmental communication. Given that the marginal cost of environmental communication is likely to be much lower than the marginal cost of greening efforts,[28] and that there is information asymmetry about companies' environmental value (Barbalau and Zeni, 2023), the greenwashing condition (9)

[28] An insightful example is the comparison of the very small certification cost of a green bond compared to the issued amount (Bank for International Settlement, 2017).

identified in our model is most likely verified. Therefore, put together, these results suggest that many companies engage in greenwashing, at least a significant portion of the time.

6 Conclusion

In this paper, we show why and how companies have an incentive to greenwash when investors have pro-environmental preferences. Companies greenwash, provided that the marginal unit cost of environmental communication is sufficiently low compared to the marginal unit cost of corporate green investments, or the information quality about their fundamental environmental value is sufficiently low. When these conditions are satisfied, companies greenwash continuously until their environmental score reaches a certain threshold above their fundamental environmental value. This threshold increases with the pro-environmental preferences of the investor, and decreases with the investor's penalty on revealed misrating.

Hence, investors can incentivize companies both to reduce the magnitude of their greenwashing effort and to increase their greening efforts by penalizing misrating revealed by controversies. This penalty, therefore, contributes to reducing the gap between environmental scores and fundamental environmental values. In addition, policymakers have complementary tools at their disposal to curb greenwashing through (i) regulations strengthening transparency on the effective environmental practices of companies and (ii) pronounced and sustained support for environment-related technological innovation to substantially reduce the marginal costs of corporate green investments. These results are robust to the introduction of interaction between companies, by assuming that investors only care about or deal with relative environmental scores. Moreover, our empirical results support the countercyclical dynamics of companies' optimal environmental communication.

Several avenues for future research naturally arise from this article. Introducing endogenous volatility would shed light on the interaction between greenwashing and corporate financial risk, at the price of additional complexity. A general equilibrium formulation would allow to consider not only investors but also consumers who are affected by greenwashing and respond by boycotting. Finally, from an empirical viewpoint, it would be valuable to estimate the dynamics of greenwashing by approximating the unknown fundamental environmental value of the companies and their greening policies.

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Internet Appendix for
“Can Investors Curb Greenwashing?”

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Abstract

In this Internet Appendix, we give a formal definition of the notion of marginal benefits (Appendix Section 1), we provide the study of two limiting cases to entail the interpretation of Proposition 2 (Appendix Section 2), we gather all the proofs of the paper in the general case (Appendix Section 3), we present the calibration used for the simulations (Appendix Section 4), we give the proofs of the model extended to the case wherein firms interact in a mean field game (Appendix Section 5), and we give the set of complementary regression tables from the empirical analysis (Appendix 6).

1 Formal definition of the marginal benefit of a strategy

To interpret the shapes of the optimal strategies, we define the notion of “marginal benefit” of each strategy. A meaningful notion of “marginal benefit” at time t in this continuous time setting can be defined as the impact on the integrated discounted cost of capital of increasing communication or greening over an infinitesimal time interval. This notion is formally defined below. In this section we fix a given company i and drop the superscript i to save space.

Definition 5 (Marginal benefit of communication and greening strategies)**.** Let us define the functional J that maps the opposite value of the expected discounted integral of the cost of capital of a company to a pair of communication (c) and greening (v) strategies, dropping the i index for better readability:

$$J(c,v) := \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left\{-\Gamma + \beta E_t^{c,v} - \alpha M_t^{c,v}\right\} dt\right].$$

For a given greening strategy v, the marginal benefit of a communication strategy c is defined as the Fréchet derivative of the functional J along its first argument, evaluated in the pair of strategies (c,v), written $D_t^c J(c,v)$. Similarly, for a given communication strategy c, the marginal benefit of a greening strategy v is defined as the Fréchet derivative of the functional J along its second argument, evaluated in the pair of strategies (c,v), and written $D_t^c J(c,v)$.

In practice, these Fréchet derivatives can be explicitly derived from Gâteaux derivatives of the functional J as follows. Let $\epsilon > 0$. For a pair of communication and greening strategies $c, v \in \mathbb{A}$ and a pair of test functions $\delta c, \delta v \in \mathbb{A}$, let us define the associated pair of modified strategies:

$$c_s^\epsilon := c_s + \epsilon \delta c_s, \qquad v_s^\epsilon := v_s + \epsilon \delta v.$$

Then, the directional (Gateaux) derivatives of J along directions δc and δv are linear, and can be expressed through Frechet derivatives D_t^c and D_t^v:

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(J(c+\epsilon\delta c, v) - J(c,v)\right) &= \mathbb{E}\left[\int_0^\infty e^{-\delta t} D_t^c J(c,v)\, \delta c_t\, dt\right], \\
\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(J(c, v+\epsilon\delta v) - J(c,v)\right) &= \mathbb{E}\left[\int_0^\infty e^{-\delta t} D_t^v J(c,v)\, \delta v_t\, dt\right].
\end{aligned}$$

2 Limiting cases

To entail the understanding and interpretation of the optimal strategy (Proposition 2), we additionally study two limiting cases: the case wherein the investor has pro-environmental preferences but does not penalize misrating ($\beta > 0$, $\alpha = 0$), and the one wherein she penalizes misrating but does not have pro-environmental preferences ($\alpha > 0$, $\beta = 0$).

2.1 Pro-environmental preferences, no misrating penalty ($\beta > 0, \alpha = 0$)

In this subsection, we assume that $\beta > 0$ and $\alpha = 0$. In this limiting case, the greening and environmental communication efforts serve the sole purpose of optimally increasing the company's environmental score, by balancing the benefit of the reduction in cost of capital enabled by these strategies against their respective financial costs.

The optimal distribution of spending between these two types of strategies depends on the degree quality of information about the fundamental environmental value of the company, characterized by the notion of "revelation intensity," as shown in the next proposition.[29]

Proposition 9 (Optimal strategies)**.** *When the investor has pro-environmental preferences only, optimal efforts of greening and environmental communication are constant, and have the following values:*

$$v_t^{i,*} = \frac{1}{\kappa_v^i}\left(\frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda^i}\right), \qquad c_t^{i,*} = \frac{1}{\kappa_c^i}\frac{\beta}{\delta + a + b\lambda^i}. \tag{2.1}$$

In the absence of penalty on misrating, the marginal benefits of communication, $\frac{\beta}{\delta+a+b\lambda^i}$, and greening, $\frac{\beta}{\delta} - \frac{\beta}{\delta+a+b\lambda^i}$, represent the benefit of increasing the environmental score through a raise in communication or greening, respectively. This benefit is constant and positive for both strategies, as it does not depend on the stochastic overrating of the company, $E_t^{i,*} - V_t^{i,*}$. The marginal benefits of the two environmental strategies depend directly, and in opposite ways, on the degree of information quality, through the revelation intensity, $a + b\lambda^i$. Indeed, a low degree of information quality makes environmental communication (greening) more (less) efficient at raising the environmental score, because, on average, its impact lasts longer (is delayed).

The optimal greenwashing strategy of company i, in this context, is given in the following proposition.

[29] Proofs of propositions in the two limiting cases are provided in Appendix 3.4.

Proposition 10 (Greenwashing effort)**.** *When condition*

$$\frac{\kappa_v^i}{\kappa_c^i} > \frac{a + b\lambda^i}{\delta} \tag{2.2}$$

is satisfied, company i engages in positive communication effort $c_t^{i,*} > 0$, which is higher than its greening effort: $c_t^{i,*} > v_t^{i,*}$. Therefore, except when the company is underrated ($E_t^{i,*} < V_t^{i,*}$) due to measurement error, it always greenwashes. Moreover, its greenwashing effort, $c_t^{i,*} - v_t^{i,*}$, is constant and equal to the positive quantity $G_i^\beta > 0$, with $G_i^\beta = \frac{2}{\bar{\kappa}^i} \frac{\beta}{\delta + a + b\lambda^i} - \frac{\beta}{\delta \kappa_v^i}$.

When condition (2.2) *is not satisfied, company i never greenwashes.*

Greenwashing practices of company i depend on the "ON-OFF" condition (2.2), which compares the ratio of marginal benefits of the two strategies, $\left(a + b\lambda^i\right)/\delta$, with their relative marginal unit costs, κ_v^i/κ_c^i: when it is sufficiently cheap to engage in environmental communication relative to greening, when the quality of information is sufficiently low, or when the company's rate of time preference is high enough, the company greenwashes. Otherwise, it never engages in greenwashing. When condition (2.2) is satisfied, the amount of greenwashing effort, G_i^β, is constant, and it (i) increases linearly in the investor's green sensitivity, β, (ii) decreases with the degree of information quality (decreases with the revelation intensity) and the marginal unit cost of greening κ_v^i, and (iii) decreases with the marginal unit cost of communication κ_c^i.

We are now able to determine the impact of company i's greenwashing effort (see Definition 4).

Proposition 11 (Greenwashing impact)**.** *When condition* (2.2) *is satisfied, the impact of company i's optimal greenwashing strategy is equal to*

$$\lim_{t \to \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = \frac{1}{a + b\lambda^i} G_i^\beta > 0,$$

where the convergence takes place with an exponential rate.

Company i's greenwashing strategy, therefore, induces a positive bias in its environmental score, which becomes, on average, higher than its environmental value. This bias increases linearly in the greenwashing effort, G_i^β, and hence, increases linearly in the investor's green sensitivity, β.

Therefore, when condition (2.2) is satisfied, a higher marginal unit cost of greening, degree of information imperfection, or rate of time preference leads to an increase in greenwashing effort and impact, while greening decreases. As for the investor's pro-environmental sensitivity, β, its increase leads to an increase in both greenwashing and greening efforts at the same rate: their ratio remains constant (Propositions 9 and 10).

2.2 Misrating penalty, no pro-environmental preferences ($\alpha > 0, \beta = 0$)

In this subsection, we assume that $\beta = 0$ and $\alpha > 0$. In this limiting case, greening effort and environmental communication of company i are solely directed towards increasing the accuracy of its environmental score, that is, to bring it closer to its environmental value.

Proposition 12 (Optimal strategies)**.** *When the representative investor does not have pro-environmental preferences ($\beta = 0$), but penalizes misrating ($\alpha > 0$), the optimal greening and communication efforts are as follows:*

$$v_t^{i,*} = \frac{A^i}{\kappa_v^i}(E_t^{i,*} - V_t^{i,*}), \qquad c_t^{i,*} = -\frac{A^i}{\kappa_c^i}(E_t^{i,*} - V_t^{i,*}), \tag{2.3}$$

with $A^i > 0$ given in Proposition 2. In this context,

(i) When the company is overrated ($E_t^{i,*} - V_t^{i,*} > 0$), it engages in brown communication ($c_t^{i,*} < 0$) and abates ($v_t^{i,*} > 0$).

(ii) When the company is underrated ($E_t^{i,*} - V_t^{i,*} < 0$), it engages in green communication ($c_t^{i,*} > 0$) and makes brown investment ($v_t^{i,*} < 0$).

The marginal benefits of greening and environmental communication, $A^i(E_t^{i,*} - V_t^{i,*})$ and $-A^i(E_t^{i,*} - V_t^{i,*})$, respectively, are now stochastic and depend on the company's overrating, $(E_t^{i,*} - V_t^{i,*})$, with the same coefficient but opposite signs. Thus, these strategies work in opposite directions and symmetrically at reducing the discrepancy between the environmental score and the fundamental environmental value of the company. For example, when the environmental score is higher than the environmental value, the company spends on greening, $v_t^{i,*} > 0$, and brown communication, $-c_t^{i,*} > 0$, until their marginal costs, $\kappa_v^i r_t^{i,*}$ and $-\kappa_c^i c_t^{i,*}$, respectively, equal their marginal benefit, $A^i(E_t^{i,*} - V_t^{i,*})$. Therefore, the coefficient A^i drives a "corrective force" and represents the expected marginal discounted penalty on the company's cost of capital when the environmental score is one unit above the environmental value.

The next proposition characterizes the optimal greenwashing policy in this second limiting case.

Proposition 13 (Greenwashing)**.** *When the investor does not have pro-environmental preferences* ($\beta = 0$)*, the companies never engage in greenwashing. Therefore, their ratings are, on average, accurate: for every company* i*,* $\lim_{t\to\infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = 0$*, where the convergence takes place with an exponential rate.*

Since the investor does not have pro-environmental preferences, the companies have no benefit in increasing their environmental scores beyond their fundamental environmental values.[30] Thus, greenwashing is suboptimal in such a case.

3 Proofs

We will use the symbol $\mathbb{H}_k^2(h)$ to denote the set of all $\mathbb{F}$-progressively measurable $\mathbb{R}^k$-valued processes $\eta = (\eta_t)_{t\in[0,T]}$ such that $\mathbb{E}[\int_0^\infty e^{-ht}\|\eta_t\|^2 dt] < \infty$ for any parameter $h \in \mathbb{R}_+^*$.

[30]See Equation (8), wherein the right-hand side is zero.

3.1 Equilibrium expected returns and optimal strategy

We restate Proposition 1 with its full set of assumptions in Proposition 14, before proving it.

Proposition 14. *Let us assume that* E, M*, solutions of dynamics* (2a) *and* (3)*, verify* $E, M \in \mathbb{H}^2_N(\delta^I)$*. Moreover, let us define* S *as a solution to* (1) *and the set of admissible strategies* $\mathbb{A}^\omega$ *for the program of the investor* (4) *as* $\mathbb{A}^\omega := \mathbb{H}^2_N(\delta^I)$*.*

Then, the optimal portfolio choice of the investor is the pointwise solution

$$\omega_t^* = \frac{1}{\gamma}\Sigma^{-1}(\mu_t + \beta E_t - \alpha M_t),$$

and equilibrium expected returns are

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta E_t + \alpha M_t.$$

Proof of Propositions 1 and 14. Under the assumptions of the proposition, the investor's program can be rewritten as

$$\begin{aligned} &\sup_{\omega \in \mathbb{A}^\omega} \mathbb{E}\left[\int_0^\infty e^{-\delta^I t}\omega_t' \left(\mu_t + \beta E_t - \alpha M_t - \frac{\gamma}{2}\Sigma\omega_t\right) dt\right] \\ &= \sup_{\omega \in \mathbb{A}^\omega} \mathbb{E}\left[\int_0^\infty e^{-\delta^I t} \left\{-\frac{\gamma}{2}(\omega_t - \omega_t^*)'\Sigma(\omega_t - \omega_t^*) + \frac{\gamma}{2}\omega_t^{*\prime}\Sigma\omega_t^*\right\} dt\right]. \end{aligned}$$

The optimal portfolio choice of the investor is thus the pointwise solution ω_t^*. In addition, as the quantity of each asset is assumed to be normalised to one in the market, writing $\mathbf{1}_n$ a vector of ones of size n, market clearing condition writes:

$$\forall t,\ \omega_t^* = \mathbf{1}_n.$$

Equilibrium expected returns are therefore

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta E_t + \alpha M_t.$$

□

Proof of Proposition 2. As the problem is symmetric for each company and depends solely on its own variables and parameters, we drop the exponent i to lighten notations throughout the proof.

The value function of the company's program is as follows:

$$\hat{w}(q,p,u) = \inf_{(v,c)\in\mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\Gamma - \beta E_t^q + \alpha M_t^p + \frac{\kappa_v}{2}(v_t)^2 + \frac{\kappa_c}{2}(c_t)^2\right)dt\right],$$

with the following constraints. The state variables of the company's program are the tridimensional process (E^q, V^p, M^u) which is the unique strong solution (Protter, 2005, Chapter 5, Theorem 52) to the following SDEs:

$$\begin{cases} dE_t^q = a(V_t^p - E_t^q)dt + (V_{t-}^p - E_{t-}^q)d\widetilde{N}_t + c_t dt + z dW_t, \quad E_0^q = q, \\ dV_t^p = v_t dt, \quad V_0^p = p, \\ dM_t^u = -\rho M_t^u dt + (V_{t-}^p - E_{t-}^q)^2 d\widehat{N}_t, \quad M_0^u = u, \end{cases} \tag{3.4}$$

for $(q,p,u) \in \mathcal{Y}$, $\mathcal{Y} := \mathbb{R}^2 \times \mathbb{R}_+$ and $(c,v) \in \mathbb{A}$. Here, $\widetilde{N}$ is a compound Poisson process with intensity λ and jump size distribution $B(1, 1/b-1)$, independent from W, and $\widehat{N}$ is a compound Poisson process such that $\Delta\widehat{N}_t = (\Delta\widetilde{N}_t)^2$. The set of admissible strategies is

$$\mathbb{A} := \left\{(c,v) \in \mathbb{H}_2^2(\delta^I \wedge \delta)\right\}.$$

Remark that, as admissible strategies $(c,v) \in \mathbb{A}$ are in $\mathbb{H}_2^2(\delta^I \wedge \delta)$, they are both in $\mathbb{H}_2^2(\delta^I)$ and $\mathbb{H}_2^2(\delta)$.

Equivalence with an auxiliary program First, remark that $\hat{w}$ is equivalent to the following program (which differs only through a constant):

$$\sup_{(v,c)\in\mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\beta E_t^q - \alpha M_t^u - \frac{\kappa_v}{2}(v_t)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right]$$

Modulo adding a constant term, and using that $e^{-\delta t}\mathbb{E}[V_t^p] \underset{t\to\infty}{\longrightarrow} 0$ according to Lemma 3.1 and 3.2 for any admissible control, this can be rewritten as:

$$\sup_{(v,c)\in\mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\beta(E_t^q - V_t^p) - \alpha M_t^u - \frac{\kappa_v}{2}\left(v_t - \frac{\beta}{\delta\kappa_v}\right)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right].$$

Then, remark that for all $t \geq 0$,

$$\frac{\kappa_v}{2}\left(v_t - \frac{\beta}{\delta\kappa_v}\right)^2 + \frac{\kappa_c}{2}(c_t)^2 = \frac{\bar{\kappa}}{4}\left(c_t - v_t + \frac{\beta}{\delta\kappa_v}\right)^2 + \frac{1}{2(\kappa_v+\kappa_c)}(\kappa_c c_t + \kappa_v v_t - \frac{\beta}{\delta})^2.$$

Let $\xi_t = c_t - v_t$ with $(v,c)\in\mathbb{A}$ and introduce the new state process $X_t = E_t^q - V_t^p$, so that

$$dX_t^x = -aX_t^x dt - X_{t-}^x d\widetilde{N}_t + \xi_t dt + z dW_t, \quad X_0 = x = q - p,$$

$$dM_t^u = -\rho M_t^u dt + (-X_{t-}^x)^2 d\widehat{N}_t, \quad M_0 = u.$$

We have $\hat{w}(q,p,u) = \tilde{w}(x,u)$, with

$$\tilde{w}(x,u) = \sup_{\substack{\xi=c-v,\\(v,c)\in\mathbb{A}}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\beta X_t^x - \alpha M_t^u - \frac{\bar{\kappa}}{4}\left(\xi_t + \frac{\beta}{\delta\kappa_v}\right)^2 - \frac{1}{2(\kappa_v+\kappa_c)}(\kappa_c c_t + \kappa_v v_t - \frac{\beta}{\delta})^2\right)dt\right].$$

It is then clear that at optimum, the controls satisfy

$$\kappa_c c_t + \kappa_v v_t - \frac{\beta}{\delta} = 0. \tag{3.5}$$

We can then parameterize the two controls with a single process $\xi_t = c_t - v_t$.

This allows us to rewrite the program as a bidimensional problem, that is, with only two state variables. Consider the auxiliary optimization problem w on $\mathcal{X} := \mathbb{R}\times\mathbb{R}_+$ as follows:

$$w(x,u) = \sup_{\xi\in\mathbb{A}^\xi} \mathbb{E}\left[\int_0^\infty e^{-\delta t} f(X_t^x, M_t^u, \xi_t)dt\right], \tag{3.6}$$

with $f(x,u,\xi) := \beta x - \alpha u - \frac{\bar{\kappa}}{4}\left(\xi + \frac{\beta}{\delta\kappa_v}\right)^2$, and the auxiliary bidimensional state variables process (X^x, M^u) as the unique strong solution to the following SDEs (Protter, 2005, Chapter 5, Theorem 52):

$$\begin{cases} dX_s^x = -aX_s ds - X_{s-} d\widetilde{N}_s + \xi_s ds + z dW_s, \quad X_0 = x, \\ dM_s^u = -\rho M_s ds + (X_{s-})^2 d\widehat{N}_s, \quad M_0 = u, \end{cases} \tag{3.7}$$

for $(x,u) \in \mathcal{X}$ and $\xi \in \mathbb{A}^\xi$ the set of admissible strategies, verifying

$$A^\xi := \left\{\xi \in \mathbb{H}_1^2(\delta^I \wedge \delta)\right\}.$$

Note that, by construction, any control (c,v) which verifies equation (3.5) verifies that

$$(c,v) \in \mathbb{A} \iff c - v \in \mathbb{A}^\xi. \tag{3.8}$$

In particular, this is true for optimal controls.

Moreover, note that for any $\xi \in \mathbb{A}^\xi$, the bidimensional auxiliary state variable (3.7) admits the following explicit solutions:

$$\begin{cases} X_t^x = \mathcal{E}_t x + \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1}\left\{\xi_s ds + z dW_s\right\} \quad \text{if} \quad 0 \le b < 1, \\ X_t^x = \mathbb{1}_{t<\theta_1}\left(x e^{-at} + \int_0^t e^{-a(t-s)}\left\{\xi_s ds + z dW_s\right\}\right) + \mathbb{1}_{t \ge \theta_1} \int_{\theta(t)}^t e^{-a(t-s)}\left\{\xi_s ds + z dW_s\right\} \quad \text{if} \quad b = 1, \end{cases} \tag{3.9}$$

$$M_t^u = e^{-\rho t}u + \int_0^t e^{-\rho(t-s)}(X_{s-}^x)^2 d\widehat{N}_s, \tag{3.10}$$

with

$$\mathcal{E}_t = e^{-at}\prod_{s \le t}(1 - \Delta\widetilde{N}_s), \qquad \theta(t) = \sup\{s \le t : d\widetilde{N}_s \neq 0\}, \qquad \theta_1 = \inf\{s \ge 0 : d\widetilde{N}_s \neq 0\}.$$

Solving the HJB equation of the auxiliary program We first show how the HJB equation satisfied by the value function w of the auxiliary problem may be solved explicitly, and then, in the next paragraph, prove a verification theorem which shows that the explicit solution found in this paragraph indeed coincides with the value function. Consider the following HJB equation: for $b < 1$,

$$\max_{\xi \in \mathbb{R}} \Big\{ \beta x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\delta \kappa_v} \right)^2 - \delta w + \frac{\partial w}{\partial x}(-ax + \xi) - \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda(1/b - 1) \int_0^1 (1-y)^{1/b-2} \left[w(x(1-y), u + y^2 x^2) - w(x,u) \right] dy \Big\} = 0, \quad (3.11)$$

and for $b = 1$,

$$\max_{\xi \in \mathbb{R}} \Big\{ \beta x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\delta \kappa_v} \right)^2 - \delta w + \frac{\partial w}{\partial x}(-ax + \xi) - \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda \left[w(0, u + x^2) - w(x,u) \right] \Big\} = 0, \quad (3.12)$$

In other words, replacing ξ by the optimizing function $\xi^*(x,u) := \frac{2}{\bar{\kappa}} \frac{\partial w}{\partial x} - \frac{\beta}{\delta \kappa_v}$, for $b < 1$,

$$\beta x - \alpha u + \frac{1}{\bar{\kappa}} \left(\frac{\partial w}{\partial x} \right)^2 - \delta w - \frac{\partial w}{\partial x}(ax + \frac{\beta}{\delta \kappa_v}) - \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda(1/b - 1) \int_0^1 (1-y)^{1/b-2} \left[w(x(1-y), u + y^2 x^2) - w(x,u) \right] dy = 0,$$

and similarly for $b = 1$. Let us use the ansatz

$$\phi(x,u) = \frac{1}{2} A x^2 + Bx + Cu + w_0.$$

Substituting this function and its derivatives $\frac{\partial \phi}{\partial x} = Ax + B$, $\frac{\partial \phi}{\partial u} = C$, $\frac{\partial^2 \phi}{\partial x^2} = A$ into the HJB equation, we get, for $b < 1$:

$$\beta x - \alpha u + \frac{1}{\bar{\kappa}} (Ax+B)^2 - \delta(\frac{1}{2}Ax^2 + Bx + Cu + w_0)$$
$$-(Ax+B)(ax+\frac{\beta}{\delta\kappa_v}) - C\rho u + \frac{z^2}{2}A + \lambda(1/b-1)\int_0^1 (1-y)^{1/b-2}\left[\frac{1}{2}Ax^2((1-y)^2-1) - Byx + Cy^2x^2\right] dy = 0.$$

Computing the integral explicitly, we get, for all $b \in [0,1]$,

$$\beta x - \alpha u + \frac{1}{\bar{\kappa}} (Ax+B)^2 - \delta(\frac{1}{2}Ax^2 + Bx + Cu + v_0)$$
$$- (Ax+B)(ax + \frac{\beta}{\delta\kappa_v}) - C\rho u + \frac{z^2}{2}A + \lambda\left[-\frac{bAx^2}{b+1} - bBx + \frac{2b^2Cx^2}{b+1}\right] = 0.$$

Collecting terms with the same powers of u and x, we get that A, B, C are characterized by the following equations:

$$-\alpha - \delta C - \rho C = 0 \tag{3.13}$$
$$\frac{2}{\bar{\kappa}}A^2 - \left(\frac{2\lambda b}{b+1} + \delta + 2a\right)A + \frac{4\lambda b^2}{b+1}C = 0 \tag{3.14}$$
$$\left(\frac{2}{\bar{\kappa}}A - \delta - a - \lambda b\right)B + \beta - A\frac{\beta}{\delta\kappa_v} = 0 \tag{3.15}$$

and the candidate optimal control is

$$\hat{\xi}_t = \xi^*(\hat{X}_t^x, \hat{M}_t^u) = \frac{2}{\bar{\kappa}}\left(A\hat{X}_t^x + B\right) - \frac{\beta}{\delta\kappa_v}$$

with $(\hat{X}_t^x, \hat{M}_t^u)$ the unique strong solutions of (3.7) when the control $\hat{\xi}$ is employed (Protter, 2005, Chapter 5, Theorem 52).

According to equations (3.13) and (3.15), C and B are given as follows:

$$C = -\frac{\alpha}{\rho+\delta}, \qquad B = \frac{\beta(\frac{A}{\delta\kappa_v} - 1)}{\left(\frac{2}{\bar{\kappa}}A - \delta - (a+\lambda b)\right)}.$$

The polynomial of degree 2 in A in equation (3.14) has two roots. One is strictly positive (> 0) (let us call it A^+), and the other one is negative ($A^- \leq 0$) (strictly negative if $\alpha > 0$), as follows:

$$A^- = \frac{\bar{\kappa}}{4}\left(\delta + 2a + \frac{2\lambda b}{1+b} - \sqrt{\left(\delta + 2a + \frac{2\lambda b}{1+b}\right)^2 + \frac{32\lambda b^2}{\bar{\kappa}(1+b)}\frac{\alpha}{\delta+\rho}}\right), \tag{3.16}$$

$$A^+ = \frac{\bar{\kappa}}{4}\left(\delta + 2a + \frac{2\lambda b}{1+b} + \sqrt{\left(\delta + 2a + \frac{2\lambda b}{1+b}\right)^2 + \frac{32\lambda b^2}{\bar{\kappa}(1+b)}\frac{\alpha}{\delta+\rho}}\right). \tag{3.17}$$

Lemma 3.7 shows that the candidate optimal control associated to A^+, $\xi_t^+ = \frac{2}{\bar{\kappa}}\left(A^+ X_t^+ + B\right) - \frac{\beta}{\delta\kappa_v}$, with X^+ the strong solution of the first SDE in (3.7) controlled by ξ^+, is not admissible. Thus, in what follows, we will write $A := -A^-$, and show that the value function of the auxiliary problem is indeed given by the solution of the HJB equation we have just found.

Verification argument for the auxiliary program Let us define on $\mathcal{X}$ the function

$$\phi(x,u) = -\frac{1}{2}Ax^2 + Bx + Cu + w_0.$$

Let us show that $w = \phi$.

(i) Let $\xi \in \mathbb{A}^\xi$. By Itô's formula applied to $e^{-\delta t}\phi(X_t^x, M_t^u)$ between 0 and the stopping time τ_n defined below, we have:

$$e^{-\delta(t\wedge\tau_n)}\phi(X_{t\wedge\tau_n}^x, M_{t\wedge\tau_n}^u) = \phi(x,u) + \int_0^{t\wedge\tau_n} e^{-\delta s}\left[-\delta\phi(X_s^x, M_s^u) + \mathcal{L}^{\xi_s}\phi(X_s^x, M_s^u)\right]ds$$
$$+ \int_0^{t\wedge\tau_n} e^{-\delta s}\frac{\partial\phi}{\partial x}(X_s^x, M_s^u)z\,dW_s,$$

with the stopping time

$$\tau_n := \inf\{t \geq 0 : \int_0^t e^{-\delta s}|\frac{\partial\phi}{\partial x}(X_s^x, M_s^u)|^2 ds \geq n\}, \quad \forall n \in \mathbb{N},$$

using the convention that $\inf\{\emptyset\} = \infty$, and the operator $\mathcal{L}^\xi \phi$ defined as follows: for $b < 1$,

$$\forall (x,u) \in \mathcal{X}, \quad \mathcal{L}^\xi \phi(x,u) := \frac{\partial \phi}{\partial x}(-ax+\xi) - \frac{\partial \phi}{\partial u}\rho u + \frac{z^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \lambda(1/b-1)\int_0^1 (1-y)^{1/b-2}\left[\phi(x(1-y), u + y^2x^2) - \phi(x,u)\right] dy,$$

and for $b = 1$,

$$\forall (x,u) \in \mathcal{X}, \quad \mathcal{L}^\xi \phi(x,u) := \frac{\partial \phi}{\partial x}(-ax+\xi) - \frac{\partial \phi}{\partial u}\rho u + \frac{z^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \lambda\left[\phi(0, u + x^2) - \phi(x,u)\right].$$

The stopped stochastic integral is a martingale, and by taking the expectation we get

$$\begin{aligned}\mathbb{E}[e^{-\delta(t\wedge\tau_n)}\phi(X^x_{t\wedge\tau_n}, M^u_{t\wedge\tau_n})] &= \phi(x,u) + \mathbb{E}[\int_0^{t\wedge\tau_n} e^{-\delta s}\left\{-\delta\phi(X^x_s, M^u_s) + \mathcal{L}^{\xi_s}\phi(X^x_s, M^u_s)\right\} ds] \\ &\le \phi(x,u) - \mathbb{E}[\int_0^{t\wedge\tau_n} e^{-\delta s} f(X^x_s, M^u_s, \xi_s) ds],\end{aligned}$$

from (3.11), as ξ is any admissible control. By Lemmas 3.4 and 3.5, we may apply the dominated convergence theorem and send n to infinity:

$$\mathbb{E}[e^{-\delta t}\phi(X^x_t, M^u_t)] \le \phi(x,u) - \mathbb{E}[\int_0^t e^{-\delta s} f(X^x_s, M^u_s, \xi_s) ds]. \tag{3.18}$$

By sending now t to infinity, using again Lemmas 3.4 and 3.5, we then deduce

$$\phi(x,u) \ge \mathbb{E}\left[\int_0^\infty e^{-\delta s} f(X^x_s, M^u_s, \xi_s) ds\right], \quad \forall \xi \in \mathbb{A}^\xi,$$

and so $\phi \ge w$ on $\mathcal{X}$.

(ii) By repeating the above arguments and observing that the optimal control $\hat{\xi}$ achieves equality in (3.18) by construction, we have

$$\mathbb{E}[e^{-\delta t}\phi(\hat{X}^x_t, \hat{M}^u_t)] = \phi(x,u) - \mathbb{E}[\int_0^t e^{-\delta s} f(\hat{X}^x_s, \hat{M}^u_s, \hat{\xi}_s) ds].$$

From Lemma 3.6, $\hat{\xi} \in \mathbb{A}^\xi$, and hence Lemma 3.5 can be applied. By sending t to infinity, we then deduce

$$\phi(x,u) \leq \mathbb{E}\left[\int_0^\infty e^{-\delta t}|f(\hat{X}_s^x, \hat{M}_s^u, \hat{\xi}_s)|ds\right] \leq w(x,u).$$

Combining with the conclusion to (i), this shows that $\phi = w$ on $\mathcal{X}$, and that the process $\{\hat{\xi}_t = \xi^*(\hat{X}_t^x, \hat{M}_t^u),\ t \geq 0\}$ is an optimal control.

Now, from Lemma 3.8, we get that that if ξ^1 and ξ^2 are both optimal controls, then

$$\int_0^\infty e^{-\delta t}|\xi_t^1 - \xi_t^2|^2 dt = 0,$$

hence the optimal control is unique, up to t-almost everywhere and almost sure equivalence.

Conclusion for the initial optimization program By (3.5) and (3.8), we can deduce the unique optimal control (c^*, v^*) to the equivalent program $\hat{w}$ from the following system:

$$\begin{cases} \kappa_c c_t^* + \kappa_v v_t^* - \frac{\beta}{\delta} = 0, \\ \xi_t^* = c_t^* - v_t^*. \end{cases}$$

Hence, optimal strategies of the company are as follows:

$$v_t^* = \frac{1}{\kappa_v}\left(A(E_t^* - V_t^*) + \frac{\beta}{\delta} - B\right), \quad c_t^* = \frac{1}{\kappa_c}\left(-A(E_t^* - V_t^*) + B\right),$$

with (E^*, V^*, M^*) solutions of (3.4) controlled by (c^*, v^*), $A = -A^-$ and

$$B = \frac{\beta(1 + \frac{A}{\delta\kappa_v})}{\frac{2}{\bar{\kappa}}A + \delta + a + \lambda b},$$

with A^- given in (3.16). As all parameters are positive, A and B are positive coefficients. Moreover,

$$\frac{\beta}{\delta} - B \geq 0 \iff \frac{1}{\delta} > \frac{1 + \frac{A}{\delta\kappa_v}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A} \iff 1 > \frac{\delta + \frac{A}{\kappa_v}}{\delta + a + \lambda b + \left(\frac{1}{\kappa_v} + \frac{1}{\kappa_c}\right)A},$$

which is always true as all parameters are non negative. ☐

Lemma 3.1. *If $\eta \in \mathbb{H}^2_1(\delta)$, then $\mathbb{E}\left[\int_0^\infty e^{-\delta t}|\eta_t|dt\right] < \infty$. Hence, in particular, $\int_0^\infty e^{-\delta t}|\eta_t|dt < \infty$ a.s..*

Proof. We have

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}|\eta_t|dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\delta t}|\eta_t|\mathbb{1}_{|\eta_t|\geq 1}dt + \int_0^\infty e^{-\delta t}|\eta_t|\mathbb{1}_{|\eta_t|<1}dt\right]$$
$$\leq \mathbb{E}\left[\int_0^\infty e^{-\delta t}|\eta_t|^2 dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\delta t}dt\right] < \infty$$

as $\eta \in \mathbb{H}^2_1(\delta)$. ☐

Lemma 3.2. *(i) Let η a progressively measurable process verifying $\int_0^\infty e^{-\delta t}|\eta_t|dt < \infty$ a.s. Then,*

$$\lim_{t\to\infty} e^{-\delta t}\int_0^t |\eta_s|ds = 0 \text{ a.s..}$$

(ii) If moreover $\mathbb{E}\left[\int_0^\infty e^{-\delta t}|\eta_t|dt\right] < \infty$, then $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[\int_0^t |\eta_s|ds] = 0$.

Proof. (i) Assume $\lim_{t\to\infty} e^{-\delta t}\int_0^t |\eta_s|ds$ does not exist or is not zero for a non null probability. Therefore, there exists a measurable set $N \subset \Omega$, $\mathbb{P}(N) > 0$, so that $\lim_{t\to\infty} e^{-\delta t}\int_0^t |\eta_s|ds$ does not exist or is nonzero for every $\omega \in N$. Let us reason for a fixed $\omega \in N$. Then, there exists $c > 0$ and an increasing sequence $(t_n) \in \mathbb{R}_+^{\mathbb{N}}$ which tends to ∞ so that $e^{-\delta t_n}\int_0^{t_n} |\eta_s|ds > c$ for every n. Take two natural numbers k, l with $k \leq l$. Define $c_1 := e^{-\delta t_k}\int_0^{t_k} |\eta_s|ds$. Then $e^{-\delta t_l}\int_0^{t_k} |\eta_s|ds = c_1 e^{-\delta(t_l - t_k)}$ and hence $e^{-\delta t_l}\int_{t_k}^{t_l} |\eta_s|ds > c - c_1 e^{-\delta(t_l-t_k)}$. When t_l is big enough, there exists $\gamma > 0$ so that $c - c_1 e^{-\delta(t_l - t_k)} > \gamma$. Moreover,

$$e^{-\delta t_l}\int_{t_k}^{t_l} |\eta_s|ds \leq \int_{t_k}^{t_l} e^{-\delta s}|\eta_s|ds.$$

So for t_l big enough, we get $\int_{t_k}^{t_l} e^{-\delta s}|\eta_s|ds \geq \gamma > 0$ with a non-null probability. Now, take t_l, t_k to ∞. As $\int_0^\infty e^{-\delta t}|\eta_t|dt$ converges almost surely, $\int_{t_k}^{t_l} e^{-\delta s}|\eta_s|ds$ tends to zero almost surely. There is a contradiction. Hence, $\lim_{t\to\infty} e^{-\delta t}\int_0^t |\eta_s|ds = 0$ a.s..

(ii) By Fubini, $\mathbb{E}\left[\int_0^\infty e^{-\delta t}|\eta_t|dt\right] < \infty$ implies that $\int_0^\infty e^{-\delta t}\mathbb{E}\left[|\eta_t|\right]dt < \infty$. By the same argument as in part (i), $\lim_{t\to\infty} e^{-\delta t}\int_0^t \mathbb{E}[|\eta_s|]ds = 0$. Applying Fubini again, conclude that $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[\int_0^t |\eta_s|ds] = 0$. □

Lemma 3.3. *If $\xi \in \mathbb{H}_1^2(\delta)$, then*

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t |X_s^x|^2 ds\right)dt\right] < \infty.$$

Moreover, $\forall t \geq 0$, $\mathbb{E}[|M_t^u|] < \infty$.

Proof. (i) By integration by parts,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t |X_s^x|^2 ds\right)dt\right] &= \mathbb{E}\left[\int_0^\infty |X_t^x|^2\left(\int_t^\infty e^{-\delta s}ds\right)dt - \lim_{t\to\infty}\left(\int_t^\infty e^{-\delta s}ds\right)\left(\int_0^t |X_s^x|^2 ds\right)\right]\\ &= \frac{1}{\delta}\mathbb{E}\left[\int_0^\infty e^{-\delta t}|X_t^x|^2 dt - \lim_{t\to\infty} e^{-\delta t}\left(\int_0^t |X_s^x|^2 ds\right)\right]\end{aligned}$$

Now, referring to the explicit expression of X^x in (3.9), we have, for $b < 1$, using Jensen inequality,

$$\begin{aligned}|X_t^x|^2 &\leq 3\left(\mathcal{E}_t^2|x|^2 + \int_0^t ds\int_0^t |\xi_s|^2(\mathcal{E}_t\mathcal{E}_s^{-1})^2 ds + z^2\left(\int_0^t \mathcal{E}_t\mathcal{E}_s^{-1}dW_s\right)^2\right)\\ &\leq 3\left(|x|^2 + \frac{1}{a}\int_0^t |\xi_s|^2 ds + z^2\left(\int_0^t \mathcal{E}_t\mathcal{E}_s^{-1}dW_s\right)^2\right).\end{aligned} \tag{3.19}$$

Noting that

$$\mathbb{E}\left[\left(\int_0^t \mathcal{E}_t\mathcal{E}_s^{-1}dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t e^{-2a(t-s)}\prod_{s\leq r\leq t:\Delta N_r\neq 0}(1-\Delta\widetilde{N}_r)^2 ds\right] \leq t, \tag{3.20}$$

we get

$$\mathbb{E}\left[|X_t^x|^2\right] \leq 3\left(|x|^2 + \frac{1}{a}\mathbb{E}\left[\int_0^t |\xi_s|^2 ds\right] + z^2 t\right). \tag{3.21}$$

Hence, applying Fubini,

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}|X_t^x|^2 dt\right] \leq \tilde{C}\mathbb{E}\left[1+\int_0^\infty e^{-\delta t}\left(\int_0^t \xi_s^2 ds\right) dt\right]$$

with a constant $\tilde{C} > 0$. By integration by parts, we have

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t \xi_s^2 ds\right) dt\right] = \frac{1}{\delta}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\xi_t^2 dt - \lim_{t\to\infty} e^{-\delta t}\left(\int_0^t \xi_s^2 ds\right)\right].$$

As $\xi \in \mathbb{A}^\xi$, by Lemma 3.2, we have $\mathbb{E}\left[\lim_{t\to\infty} e^{-\delta t}\left(\int_0^t \xi_s^2 ds\right)\right] = 0$. Therefore,

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t \xi_s^2 ds\right) dt\right] = \frac{1}{\delta}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\xi_t^2 dt\right] < \infty.$$

Hence, we obtain that

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}|X_t^x|^2 dt\right] < \infty.$$

Using Lemma 3.2 again, this implies in particular that

$$\mathbb{E}\left[\lim_{t\to\infty} e^{-\delta t}\left(\int_0^t |X_s^x|^2 ds\right)\right] = 0.$$

As a consequence,

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t |X_s^x|^2 ds\right) dt\right] = \frac{1}{\delta}\mathbb{E}\left[\int_0^\infty e^{-\delta t}|X_t^x|^2 dt\right] < \infty$$

The same arguments can be used for $b = 1$. This concludes the first part of the proof.

(ii) Using the explicit expression of M in (3.10), we have

$$\begin{aligned}\mathbb{E}\left[|M_t^u|\right] &\leq e^{-\rho t}u + \mathbb{E}\left[\int_0^t e^{-\rho(t-s)}(X_s^x)^2 d\widehat{N}_s\right] \\ &\leq u + \frac{2\lambda b^2}{1+b}\int_0^t \mathbb{E}\left[(X_s^x)^2\right] ds.\end{aligned}$$

Now, by Fubini, $\int_0^\infty e^{-\delta t}\left(\int_0^t \mathbb{E}\left[(X_s^x)^2\right] ds\right) dt = \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t X_s^2 ds\right) dt\right]$, which is finite for $\xi \in \mathbb{A}^\xi$ according to (i). Thus, $\int_0^\infty e^{-\delta t}\left(\int_0^t \mathbb{E}\left[(X_s^x)^2\right] ds\right) dt$ is finite. By the property of the Lebesgue integral, it implies that $\int_0^t \mathbb{E}\left[(X_s^x)^2\right] ds$ is finite for t almost everywhere. Since $t \mapsto \int_0^t \mathbb{E}\left[(X_s^x)^2\right] ds$ is increasing, $\int_0^t \mathbb{E}\left[(X_s^x)^2\right] ds$ is actually finite for all $t \geq 0$, otherwise a contradiction can be easily exhibited. Hence, for all $t \geq 0$, $\mathbb{E}\left[|M_t^u|\right] < \infty$. This concludes the proof.

□

Lemma 3.4. *For any admissible control $\xi \in \mathbb{A}^\xi$, for all $(x, u) \in \mathcal{X}$, we have*

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}|f(X_t^x, M_t^u, \xi_t)|dt\right] < \infty.$$

Proof.

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}|f(X_t^x, M_t^u, \xi_t)|dt\right] \leq \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\beta|X_t^x| + \alpha M_t^u + \frac{\bar{\kappa}}{4}\left(\xi_t + \frac{\beta}{\delta\kappa_v}\right)^2\right) dt\right]$$

We have, for $b < 1$,

$$\begin{aligned} |X_t^x| &\leq \mathcal{E}_t|x| + \mathcal{E}_t|\int_0^t \mathcal{E}_s^{-1}\left\{\xi_s ds + z dW_s\right\}| \\ &\leq |x| + \int_0^t |\xi_s| ds + z\left(1 + \left(\int_0^t \mathcal{E}_t\mathcal{E}_s^{-1} dW_s\right)^2\right). \end{aligned} \tag{3.22}$$

By (3.20), we deduce

$$\mathbb{E}\left[|X_t^x|\right] \leq \mathbb{E}\left[|x| + \int_0^t |\xi_s| ds + z(1+t)\right] \tag{3.23}$$

Moreover we have, by integration by parts:

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t |\xi_s| ds\right) dt\right] = \mathbb{E}\left[\frac{1}{\delta}\left(\int_0^\infty e^{-\delta t}|\xi_t| dt + \lim_{t\to\infty} e^{-\delta t}\int_0^t |\xi_s| ds\right)\right].$$

As $\xi \in \mathbb{A}^{\xi}$, the expectation of the left term of the sum is finite by Lemma 3.1. Moreover, by Lemma 3.2, the expectation of the "lim" term is null. Applying Fubini, we finally get that

$$\mathbb{E}\left[\int_0^{\infty} e^{-\delta t}\beta|X_t^x|dt\right] < \infty.$$

The method with $b = 1$ follows the same argument.

As for $\mathbb{E}\left[\int_0^{\infty} e^{-\delta t}|M_t^u|dt\right]$,

$$\begin{aligned}\mathbb{E}\left[\int_0^{\infty} e^{-\delta t}|M_t^u|dt\right] &\leq \mathbb{E}\left[\int_0^{\infty} e^{-\delta t}\left(e^{-\rho t}u + \int_0^t e^{-\rho(t-s)}(X_s^x)^2 d\widehat{N}_s\right)dt\right] \\ &\leq \int_0^{\infty} e^{-\delta t}\left(e^{-\rho t}u + \frac{2\lambda b^2}{1+b}\int_0^t \mathbb{E}\left[(X_s^x)^2\right]ds\right)dt,\end{aligned}$$

which is finite for $\xi \in \mathbb{A}^{\xi}$ according to Lemma 3.3.

Finally, $\mathbb{E}\left[\int_0^{\infty} e^{-\delta t}\frac{\bar{\kappa}}{4}\left(\xi_t + \frac{\beta}{\delta\kappa_v}\right)^2 dt\right] \leq \mathbb{E}\left[\int_0^{\infty} e^{-\delta t}\frac{\bar{\kappa}}{2}\left(\xi_t^2 + \left(\frac{\beta}{\delta\kappa_v}\right)^2\right)dt\right]$ which is finite as $\xi \in \mathbb{H}_1^2(\delta)$. □

Lemma 3.5. *For every $\xi \in \mathbb{A}^{\xi}$ and every $t > 0$,*

$$\mathbb{E}[\sup_{0\leq s\leq t}|\phi(X_s^x, M_s^u)|] < \infty.$$

Moreover,

$$\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[\phi(X_t^x, M_t^u)\right] = 0.$$

Proof. (i) Let us show that $\mathbb{E}[\sup_{0\leq s\leq t}|\phi(X_s^x, M_s^u)|] < \infty$. We have

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|\phi(X_s^x, M_s^u)|\right] \leq \frac{1}{2}A\mathbb{E}\left[\sup_{0\leq s\leq t}(X_s^x)^2\right] + B\mathbb{E}\left[\sup_{0\leq s\leq t}|X_s^x|\right] + |C|\left(u + \mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^s e^{-\rho(s-y)}(X_y^x)^2 d\widehat{N}_y\right]\right).$$

If $b < 1$, referring to (3.19) and using Burkholder-Davis-Gundy inequality, there exists a positive constant $\tilde{C}$ so that for every $t \geq 0$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|X_s^x|^2\right] \leq \tilde{C}\mathbb{E}\left[|x|^2 + \int_0^t |\xi_u|^2 du + z^2 t\right].$$

This upper boundary is finite as $\mathbb{E}[\int_0^t |\xi_u|^2 du] \leq \mathbb{E}[e^{\delta t} \int_0^t e^{-\delta u} |\xi_u|^2 du]$, which is finite as $\xi \in \mathbb{H}_1^2(\delta)$. Thus, $\mathbb{E}\left[\sup_{0\leq s\leq t} |X_s^x|^2\right]$ is finite.

Moreover, recalling (3.22), and applying again Burkholder-Davis-Gundy inequality, we similarly get that $\mathbb{E}\left[\sup_{0\leq s\leq t} |X_s^x|\right] < \infty$ for $\xi \in \mathbb{A}^\xi$, using this time Lemma 3.1 to say that $\mathbb{E}[\int_0^\infty e^{-\delta u} |\xi_u| du] < \infty$.

Finally, as $s \mapsto \int_0^s e^{\rho y} (X_y^x)^2 d\widehat{N}_y$ is increasing for each trajectory, we have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \int_0^s e^{-\rho(s-y)} (X_y^x)^2 d\widehat{N}_y\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t} \int_0^s e^{\rho y} (X_y^x)^2 d\widehat{N}_y\right] \leq \frac{\lambda b^2}{1+b} \mathbb{E}\left[\int_0^t e^{\rho y} (X_y^x)^2 dy\right],$$

which is finite since M is integrable for admissible strategies by Lemma 3.3.

The same reasoning can be applied when $b = 1$. Therefore, we can conclude by a finite sum of finite terms that, for $0 \leq b \leq 1$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |\phi(X_s^x, M_s^u)|\right] < \infty.$$

(ii) Now, let us show that $\lim_{t\to\infty} e^{-\delta t} \mathbb{E}\left[\phi(X_t^x, M_t^u)\right] = 0$. We have

$$\begin{aligned}
\lim_{t\to\infty} |e^{-\delta t} \mathbb{E}\left[\phi(X_t^x, M_t^u)\right]| &\leq \lim_{t\to\infty} e^{-\delta t} \mathbb{E}\left[\frac{1}{2} A (X_t^x)^2 + B|X_t^x| + |C| \int_0^t e^{-\rho(t-s)} (X_s^x)^2 d\widehat{N}_s\right] \\
&= \lim_{t\to\infty} e^{-\delta t} \mathbb{E}\left[\frac{1}{2} A (X_t^x)^2 + B|X_t^x| + |C| \frac{b^2 \lambda}{1+b} \int_0^t (X_s^x)^2 ds\right]
\end{aligned}$$

using the explicit expression of M^u in (3.10), and as M^u is integrable for admissible strategies by Lemma 3.3. Again, assume $b < 1$. Since, by (3.21),

$$e^{-\delta t} \mathbb{E}\left[|X_t^x|^2\right] \leq 3 e^{-\delta t} \left(|x|^2 + \frac{1}{a} \mathbb{E}\left[\int_0^t |\xi_s|^2 ds\right] + z^2 t\right),$$

and by Lemma 3.2, $e^{-\delta t} \mathbb{E}\left[\int_0^t |\xi_s|^2 ds\right] \to 0$, (as $\xi \in \mathbb{A}^\xi$), we conclude that $\lim_{t\to\infty} e^{-\delta t} \mathbb{E}\left[\frac{1}{2} A (X_t^x)^2\right] = 0$.

Now, let us deal with $e^{-\delta t}\mathbb{E}[|X_t|]$. Similarly, by (3.23),

$$e^{-\delta t}\mathbb{E}\left[|X_t^x|\right] \leq e^{-\delta t}\left(|x| + \mathbb{E}\left[\int_0^t |\xi_s| ds\right] + z\,(1+t)\right).$$

Moreover, $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[\int_0^t |\xi_s| ds\right] = 0$ by applying successively Lemma 3.1 and 3.2, as $\xi \in \mathbb{A}^\xi$. Therefore, $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[B|X_t^x|\right] = 0$.

Finally, as $\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t (X_s^x)^2 ds\right) dt\right] < \infty$ for admissible strategies (belonging to $\mathbb{A}^\xi$) according to Lemma 3.3, it implies in particular, applying Fubini and due to the property of an infinite integral with positive integrand,

$$\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[\left(\int_0^t (X_s^x)^2 ds\right)\right] = 0.$$

The method is the same for $b = 1$. This concludes the proof.

□

Lemma 3.6. *The optimal control is admissible, i.e.* $\hat{\xi} \in \mathbb{A}^\xi$.

Proof. As $\hat{\xi}_t = \frac{2}{\bar{\kappa}}\left(-A\hat{X}_t^x + B\right) - \frac{\beta}{\delta\kappa_v}$, $\hat{X}^x, \hat{M}^u$ are solutions to the following SDEs:

$$\begin{cases} dX_s = \left(-\zeta X_s + \nu\right) ds - X_{s-} d\widetilde{N}_s + z dW_s, \quad X_0 = x, \\ dM_s = -\rho M_s ds + (X_{s-})^2 d\widehat{N}_s, \quad M_0 = u, \end{cases}$$

with $\zeta := a + \frac{2}{\bar{\kappa}}A$, $\nu := \frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v}$.

The explicit solutions for $\hat{X}^x, \hat{M}^u$ are therefore as follows:

$$\begin{cases} \hat{X}_t^x = \hat{\mathcal{E}}_t x + \hat{\mathcal{E}}_t \int_0^t \hat{\mathcal{E}}_s^{-1}\left\{\nu ds + z dW_s\right\} \quad \text{if} \quad 0 \leq b < 1, \\ \hat{X}_t^x = \mathbb{1}_{t<\theta_1}\left(e^{-\zeta t}x + \int_0^t e^{-\zeta(t-s)}\left\{\nu ds + z dW_s\right\}\right) + \mathbb{1}_{t\geq\theta_1}\int_{\theta(t)}^t e^{-\zeta(t-s)}\left\{\nu ds + z dW_s\right\} \quad \text{if} \quad b = 1, \end{cases} \tag{3.24}$$

$$\hat{M}_t^u = e^{-\rho t}u + \int_0^t e^{-\rho(t-s)}(\hat{X}_{s-}^x)^2 d\widehat{N}_s, \tag{3.25}$$

with $\theta(t), \theta_1$ defined in (3.9) and

$$\hat{\mathcal{E}}_t = e^{-\zeta t} \prod_{s \le t} (1 - \Delta \widetilde{N}_s).$$

Let us show that $\hat{\xi} \in \mathbb{H}^2_1(\delta^I \wedge \delta)$. We show it for $b < 1$. The method is the same for $b = 1$. For $b < 1$, using the explicit expression of $\hat{X}^x$ in (3.24), we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-(\delta^I \wedge \delta)t} |\hat{\xi}_t|^2 dt\right] &\le \tilde{C}\left(1 + \mathbb{E}\left[\int_0^\infty e^{-(\delta^I \wedge \delta)t} |\hat{X}^x_t|^2 dt\right]\right) \\
&\le \tilde{C}\left(1 + \mathbb{E}\left[\int_0^\infty e^{-(\delta^I \wedge \delta)t}\left(|x|^2 + \int_0^t \nu^2 ds + z^2\left(\int_0^t \hat{\mathcal{E}}_t \hat{\mathcal{E}}_s^{-1} dW_s\right)^2\right) dt\right]\right) \\
&\le \tilde{C}\left(1 + \mathbb{E}\left[\int_0^\infty e^{-(\delta^I \wedge \delta)t}\left(|x|^2 + t\nu^2 + z^2 t\right) dt\right]\right) < \infty,
\end{aligned}$$

with a positive constant $\tilde{C}$, and using (3.19) and (3.20) along with Fubini. This concludes the proof.

□

Lemma 3.7. *Let ξ^+ be the strategy defined by $\xi^+ := \frac{2}{\bar{\kappa}}\left(A^+ X^+_t + B\right) - \frac{\beta}{\delta \kappa_v}$, with A^+ given in* (3.17) *and X^+ the strong solution of the first SDE in* (3.7) *controlled by ξ^+. ξ^+ is not an admissible strategy, i.e. $\xi^+ \notin \mathbb{A}^\xi$.*

Proof. In part (ii) of the proof of Lemma 3.5, we show that, if $\xi \in \mathbb{A}^\xi$, then $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[B|X^x_t|\right] = 0$, for any $x \in \mathbb{R}$. As $|\mathbb{E}\left[X^x_t\right]| \le \mathbb{E}\left[|X^x_t|\right]$, this implies that $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[X^x_t\right] = 0$, for any $x \in \mathbb{R}$.

Let us show that $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}\left[X^+_t\right] = 0$ is not true for every initial condition $x \in \mathbb{R}$ (we do not write explicitly the dependence of X^+ on its initial condition in its exponent to lighten notations). X^+ is solution to the following SDE:

$$dX^+_s = \left(\left(\frac{2}{\bar{\kappa}} A^+ - a\right) X^+_s + \nu\right) ds - X^+_{s-} d\widetilde{N}_s + z dW_s, \quad X^+_0 = x,$$

with $\nu = \frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta \kappa_v}$. Hence, its expectation verifies the following ODE:

$$d\mathbb{E}[X^+_t] = \left(\left(\frac{2}{\bar{\kappa}} A^+ - a - b\lambda\right) \mathbb{E}[X^+_t] + \nu\right) dt.$$

This ODE has a unique solution which is, writing $\zeta^+ := \frac{2}{\bar\kappa}A^+ - a - b\lambda$,

$$\mathbb{E}[X_t^+] = e^{\zeta^+ t}\left(x + \frac{\nu}{\zeta^+}\right) - \frac{\nu}{\zeta^+}.$$

Now, remark that

$$\zeta^+ - \delta = \frac{1}{2}\left(\sqrt{\left(\delta + 2a + \frac{2\lambda b}{1+b}\right)^2 + \frac{32\lambda b^2}{\bar\kappa(1+b)}\frac{\alpha}{\delta+\rho}} - \delta - \frac{2\lambda b}{1+b}\right) \geq a > 0.$$

Hence,

$$\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[X_t^+] = \lim_{t\to\infty} e^{(\zeta^+-\delta)t}\left(x + \frac{\nu}{\zeta^+}\right) - e^{-\delta t}\frac{\nu}{\zeta^+} = \pm\infty \quad \text{if} \quad x \neq -\frac{\nu}{\zeta^+}.$$

Therefore, ξ^+ can not be an admissible strategy.

□

Lemma 3.8. *For every $\xi \in \mathbb{A}^\xi$, $\forall (x,u) \in \mathcal{X}$, the functional*

$$\mathcal{J} : (x,u,\xi) \mapsto \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(-f(X_t^x, M_t^u, \xi_t)\right)dt\right]$$

is strictly convex in ξ and for $\theta \in [0,1]$, and for $\xi^1, \xi^2 \in \mathbb{A}^\xi$,

$$\theta\mathcal{J}(x,u,\xi_1) + (1-\theta)\mathcal{J}(x,u,\xi_2) - \mathcal{J}(x,u,\theta\xi_1 + (1-\theta)\xi_2) \geq \frac{\bar\kappa}{4}\int_0^\infty e^{-\delta t}|\xi_t^1 - \xi_t^2|^2 dt.$$

Proof. Let us show that $\forall t \geq 0$, $\mathbb{E}[-f(X_t^x, M_t^u, \xi_t)]$ is convex in ξ. By linearity of integrals and applying Fubini thanks to Lemma 3.4, it will be so for $\mathcal{J}$. We first deal with the case $b < 1$. We have

$$\mathbb{E}\left[-f(X_t^x, M_t^u, \xi_t)\right] = -\beta\mathbb{E}\left[X_t^x\right] + \alpha\mathbb{E}\left[M_t^u\right] + \mathbb{E}\left[\frac{\bar\kappa}{4}\left(\xi_t + \frac{\beta}{\delta\kappa_v}\right)^2\right].$$

X_t^x is linear in ξ according to its explicit expression (3.9). The last term is obviously strictly convex in ξ_t. As for M_t^u, using its explicit expression (3.10) and the properties of admissible strategies ($\in \mathbb{A}^\xi$),

$$\mathbb{E}[M_t^u] = e^{-\rho t}u + \frac{2\lambda b^2}{b+1}\mathbb{E}\left[\int_0^t e^{-\rho(t-s)}(X_s^x)^2 ds\right].$$

Now, $(X_s^x)^2$ is strictly convex in ξ by Jensen inequality. Therefore, by addition of linear and strictly convex terms in ξ, $\mathbb{E}[-f(X_t^x, M_t^u, \xi_t)]$ is strictly convex in ξ, and so is $\mathcal{J}$. More precisely, by focusing only on the third part, it is easy to show that for $\theta \in [0,1]$, and for $\xi^1, \xi^2 \in \mathbb{A}^\xi$,

$$\theta\mathcal{J}(x,u,\xi_1) + (1-\theta)\mathcal{J}(x,u,\xi_2) - \mathcal{J}(x,u,\theta\xi_1 + (1-\theta)\xi_2) \geq \frac{\bar{\kappa}}{4}\int_0^\infty e^{-\delta t}|\xi_t^1 - \xi_t^2|^2 dt.$$

□

3.2 Marginal benefit of a strategy

Proof of Proposition 3. (i) Let us start with the marginal benefit of communication

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(J(c+\epsilon\delta c, v) - J(c,v)\right) &= \lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left\{\beta\frac{1}{\epsilon}\left(E_t^{c+\epsilon\delta c,v} - E_t^{c,v}\right) - \alpha\frac{1}{\epsilon}\left(M_t^{c+\epsilon\delta c,v} - M_t^{c,v}\right)\right\} dt\right] \\ &= \lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left\{\beta\frac{1}{\epsilon}\left(X_t^{c+\epsilon\delta c,v} - X_t^{c,v}\right) - \alpha\frac{1}{\epsilon}\left(M_t^{c+\epsilon\delta c,v} - M_t^{c,v}\right)\right\} dt\right],\end{aligned}$$

as $V_t^{c+\epsilon\delta c,v} - V_t^{c,v} = 0$.

Using the explicit expression of X (3.9), as $\xi = c - v$, we have, when $b < 1$, and for any $\epsilon > 0$,

$$\frac{1}{\epsilon}\left(X_t^{c+\epsilon\delta c,v} - X_t^{c,v}\right) = \mathcal{E}_t\int_0^t \mathcal{E}_s^{-1}\delta c_s ds.$$

Therefore, by integration by parts,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\beta\frac{1}{\epsilon}\left(X_t^{c+\epsilon\delta c,v}-X_t^{c,v}\right)dt\right]\\
&\quad=\beta\mathbb{E}\left[\int_0^\infty e^{-\delta t}\mathcal{E}_t\int_0^t \mathcal{E}_s^{-1}\delta c_s\,ds\,dt\right]\\
&\quad=\beta\mathbb{E}\left[\int_0^\infty \mathcal{E}_t^{-1}\delta c_t\int_t^\infty e^{-\delta s}\mathcal{E}_s\,dsdt-\lim_{t\to\infty}\left(\int_0^t \mathcal{E}_s^{-1}\delta c_s ds\right)\left(\int_t^\infty e^{-\delta s}\mathcal{E}_s ds\right)\right]
\end{aligned}$$

Now, using first that $\mathcal{E}_s$ is decreasing, and then Lemma 3.2 which can be applied as δc is a test function (assumed to be admissible),

$$\begin{aligned}
\left|\left(\int_0^t \mathcal{E}_s^{-1}\delta c_s ds\right)\left(\int_t^\infty e^{-\delta s}\mathcal{E}_s ds\right)\right| &\leq \left(\int_0^t \mathcal{E}_s^{-1}|\delta c_s|ds\right)\left(\int_t^\infty e^{-\delta s}\mathcal{E}_s ds\right)\\
&\leq \mathcal{E}_t\left(\int_0^t |\delta c_s|ds\right)\left(\int_t^\infty e^{-\delta s}ds\right)\mathcal{E}_t^{-1}\\
&=\frac{1}{\delta}e^{-\delta t}\left(\int_0^t |\delta c_s|ds\right)\xrightarrow{t\to\infty} 0\ a.s.
\end{aligned}$$

and therefore

$$\left|\mathbb{E}\left[\lim_{t\to\infty}\left(\int_0^t \mathcal{E}_s^{-1}\delta c_s ds\right)\left(\int_t^\infty e^{-\delta s}\mathcal{E}_s ds\right)\right]\right| \leq \mathbb{E}\left[\lim_{t\to\infty}\frac{1}{\delta}e^{-\delta t}\left(\int_0^t |\delta c_s|ds\right)\right],$$

which equals 0 by Lemma 3.2, as δc is a test function (assumed to be admissible). Hence,

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\beta\frac{1}{\epsilon}\left(X_t^{c+\epsilon\delta c,v}-X_t^{c,v}\right)dt\right]=\beta\mathbb{E}\left[\int_0^\infty e^{-\delta t}\delta c_t\,\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}\mathcal{E}_t^{-1}\mathcal{E}_s\,ds\Big|\mathcal{F}_t\right]dt\right].$$

As a consequence, the part of the Frechet derivative that is due to the X term is given by

$$\beta\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}\mathcal{E}_t^{-1}\mathcal{E}_s\,ds\Big|\mathcal{F}_t\right]=\frac{\beta}{a+\delta+b\lambda}.$$

Similar computations can be made when $b=1$, leading to the same result.

As for the terms in M, using its explicit expression (3.10), we get, using the fact that δc is admissible,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(M_t^{c+\epsilon\delta c,v}-M_t^{c,v}\right)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t e^{-\rho(t-s)}\left\{(X_{s-}^{c+\epsilon\delta c,v})^2-(X_{s-}^{c,v})^2\right\}d\widehat{N}_s$$
$$=2\int_0^t e^{-\rho(t-s)}X_{s-}^{c,v}\mathcal{E}_{s-}\left(\int_0^s \mathcal{E}_y^{-1}\delta c_y\,dy\right)d\widehat{N}_s$$

Therefore, by integration by parts, using that $\mathbb{E}\left[\int_0^\infty e^{-\delta t}M_t ds\right]<\infty$ for admissible strategies as proved in Lemma 3.4,

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\alpha\frac{1}{\epsilon}\left(M_t^{c+\epsilon\delta c,v}-M_t^{c,v}\right)dt\right]=2\alpha\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t e^{-\rho(t-s)}X_{s-}^{c,v}\mathcal{E}_{s-}\left(\int_0^s \mathcal{E}_y^{-1}\delta c_y\,dy\right)d\widehat{N}_s\right)dt\right]$$
$$=2\frac{\alpha}{\delta+\rho}\mathbb{E}\left[\int_0^\infty e^{-\delta s}X_{s-}^{c,v}\mathcal{E}_{s-}\left(\int_0^s \mathcal{E}_y^{-1}\delta c_y\,dy\right)d\widehat{N}_s\right]$$
$$=\frac{4b^2\alpha\lambda}{(\delta+\rho)(1+b)}\mathbb{E}\left[\int_0^\infty e^{-\delta s}X_s^{c,v}\mathcal{E}_s\left(\int_0^s \mathcal{E}_y^{-1}\delta c_y\,dy\right)ds\right]$$
$$=\frac{4b^2\alpha\lambda}{(\delta+\rho)(1+b)}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\delta c_t\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}X_s^{c,v}\mathcal{E}_s\mathcal{E}_t^{-1}ds\middle|\mathcal{F}_t\right]dt\right].$$

Indeed, in a similar fashion as for the X term, it can be shown that

$$\mathbb{E}\left[\lim_{t\to\infty}\left(\int_t^\infty e^{-\delta s}ds\right)\left(\int_0^t e^{-\rho(t-s)}X_{s-}^{c,v}\mathcal{E}_{s-}\left(\int_0^s \mathcal{E}_y^{-1}\delta c_y\,dy\right)d\widehat{N}_s\right)\right]=0.$$

Hence, the part of the Frechet derivative that is due to the M term is given by

$$-\frac{4b^2\alpha\lambda}{(\delta+\rho)(1+b)}\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}X_s^{c,v}\mathcal{E}_s\mathcal{E}_t^{-1}ds\middle|\mathcal{F}_t\right]$$

Joining together the X term and the M term, we finally obtain:

$$D_t^c J(c,v)=\frac{\beta}{a+\delta+b\lambda}-\frac{4b^2\alpha\lambda}{(\delta+\rho)(1+b)}\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}X_s^{c,v}\mathcal{E}_s\mathcal{E}_t^{-1}ds\middle|\mathcal{F}_t\right].$$

(ii) As for D^v_t,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(J(c,v+\epsilon\delta v)-J(c,v)\right)\\
&=\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left\{\beta\frac{1}{\epsilon}\left(E_t^{c,v+\epsilon\delta v}-E_t^{c,v}\right)-\alpha\frac{1}{\epsilon}\left(M_t^{c,v+\epsilon\delta v}-M_t^{c,v}\right)\right\}dt\right]\\
&=\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left\{\beta\frac{1}{\epsilon}\left(X_t^{c,v+\epsilon\delta v}-X_t^{c,v}\right)+\beta\frac{1}{\epsilon}\left(V_t^{c,v+\epsilon\delta v}-V_t^{c,v}\right)-\alpha\frac{1}{\epsilon}\left(M_t^{c,v+\epsilon\delta v}-M_t^{c,v}\right)\right\}dt\right],
\end{aligned}$$

Using the explicit expression of X (3.9), as $\xi = c - v$, we have, when $b<1$, and for any $\epsilon>0$

$$\frac{1}{\epsilon}\left(X_t^{c,v+\epsilon\delta v}-X_t^{c,v}\right)=-\mathcal{E}_t\int_0^t\mathcal{E}_s^{-1}\delta v_s ds.$$

Similarly to the Gateaux derivative of c, the part of the Frechet derivative that is due to the X term is given by

$$-\beta\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}\mathcal{E}_t^{-1}\mathcal{E}_s\,ds\Big|\mathcal{F}_t\right]=-\frac{\beta}{a+\delta+b\lambda}$$

The term in V is immediate, using the explicit expression of V, $V_t = p+\int_0^t v_s ds$:

$$\frac{1}{\epsilon}\left(V_t^{c,v+\epsilon\delta v}-V_t^{c,v}\right)=\int_0^t\delta v_s ds.$$

Therefore, by integration by parts, similarly to the treatment of the X term in (i),

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\beta\frac{1}{\epsilon}\left(V_t^{c,v+\epsilon\delta v}-V_t^{c,v}\right)dt\right]&=\beta\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\int_0^t\delta v_s ds\right)dt\right]\\
&=\frac{\beta}{\delta}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\delta v_t dt\right],
\end{aligned}$$

so that the part of the Frechet derivative that is due to the V term is given by $\frac{\beta}{\delta}$.

Finally, using the explicit expression of M in (3.10), we get, using the admissibility of δv,

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(M_t^{c+\epsilon\delta c,v}-M_t^{c,v}\right)&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t e^{-\rho(t-s)}\left\{(X_{s-}^{c+\epsilon\delta c,v})^2-(X_{s-}^{c,v})^2\right\}d\widehat{N}_s\\
&=-2\int_0^t e^{-\rho(t-s)}X_{s-}^{c,v}\mathcal{E}_{s-}\left(\int_0^s\mathcal{E}_y^{-1}\delta c_y\,dy\right)d\widehat{N}_s.
\end{aligned}$$

By computations similar to the case of δc, the part of the Frechet derivative that is due to the M term is given by

$$\frac{4b^2\alpha\lambda}{(\delta+\rho)(1+b)}\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}X_s^{c,v}\mathcal{E}_s\mathcal{E}_t^{-1}ds\Big|\mathcal{F}_t\right]$$

Joining together the X term, the V term and the M term, we finally obtain:

$$D_t^v J(c,v) = \frac{\beta}{\delta} - \frac{\beta}{a+\delta+b\lambda} + \frac{4b^2\alpha\lambda}{(\delta+\rho)(1+b)}\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}X_s^{c,v}\mathcal{E}_s\mathcal{E}_t^{-1}ds\Big|\mathcal{F}_t\right].$$

(iii) In view of the form of the company's optimization functional (5), the optimal communication and greening strategies c^* and v^* equalize marginal benefits and marginal costs:

$$D_t^c J(c^*,v^*) = \kappa_c c_t^*, \qquad D_t^v J(c^*,v^*) = \kappa_v v_t^*.$$

□

3.3 Interpretation of the optimal strategy

Proof of Proposition 4. The marginal rate of substitution from greening to environmental communication, $MRS^{v\to c,i}$, is as follows:

$$\mathrm{MRS}_t^{v\to c,i} = \frac{\Pi_t^{v^i,i}}{\Pi_t^{c^i,i}},$$

according to Definition 2.

When $\alpha = 0$, we have $T^i = 0$, and hence

$$\Pi_t^{v^i,i} = \frac{\beta}{\delta} - \frac{\beta}{\delta+a+b\lambda^i}, \quad \Pi_t^{c^i,i} = \frac{\beta}{\delta+a+b\lambda^i}.$$

A quick computation gives the result.

When $\beta = 0$, we get

$$\Pi_t^{v^i,i} = 2T^i\mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}\mathcal{E}_s\mathcal{E}_t^{-1}\left(E_s^i - V_s^i\right)ds\Big|\mathcal{F}_t\right],$$

$$\Pi_t^{c^i,i} = -2T^i \mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)}\mathcal{E}_s\mathcal{E}_t^{-1}\left(E_s^i - V_s^i\right)ds\Big|\mathcal{F}_t\right]$$

from Proposition 3. Hence the second result. □

Proof of Proposition 5. We drop the i indices in the proof for simplicity. We have

$$c_t^* - v_t^* = \frac{2}{\bar{\kappa}}\left(-A(E_t^* - V_t^*) + B\right) - \frac{\beta}{\delta\kappa_v}. \tag{3.26}$$

Hence, when $E_t^* \geq V_t^*$, the maximum value of $c_t^* - v_t^*$ is equal to $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v}$. Now,

$$\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v} > 0 \iff \frac{\kappa_v}{\kappa_c} > \frac{a + b\lambda}{\delta}.$$

Therefore, referring to Definition 1, if $\frac{\kappa_v}{\kappa_c} \leq \frac{a+b\lambda}{\delta}$, the company never greenwashes.

Then, we have $c_t^* > 0 \iff E_t^* - V_t^* < \frac{B}{A}$ according to the optimal communication strategy given in Proposition 2. Moreover, one can deduce out of equation (3.26) that

$$c_t^* - v_t^* > 0 \iff E_t^* - V_t^* < \frac{1}{\frac{2}{\bar{\kappa}}A}\left(\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v}\right) = \frac{B}{A} - \frac{1}{\frac{2}{\bar{\kappa}}}\frac{\beta}{\delta\kappa_v} < \frac{B}{A}.$$

Combining the two conditions and referring to Definition 1, the company greenwashes if, and only if, $0 \leq E_t^* - V_t^* < \frac{1}{\frac{2}{\bar{\kappa}}A}\left(\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v}\right)$, which is a non-empty event only under condition (9) which guarantees that $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v} > 0$, as stated above.

Moreover, as stated above, when $\frac{\kappa_v}{\kappa_c} > \frac{a+b\lambda}{\delta}$, $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v} > 0$ and it is the maximal value of $c_t^* - v_t^*$ when $E_t^* \geq V_t^*$, hence the maximal value of greenwashing effort. Moreover, greenwashing effort $c_t^* - v_t^*$ decreases linearly in $E_t^* - V_t^*$ according to equation (3.26). Finally, when $E_t^* - V_t^* = \frac{1}{\frac{2}{\bar{\kappa}}A}\left(\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_v}\right)$, greenwashing effort is null, as can be derived from the same equation.

□

Proof of Proposition 6. Let us compute $\lim_{t\to\infty}\mathbb{E}[E_t^* - V_t^*]$. We have

$$c^* - v_t^* = \frac{2}{\bar{\kappa}}\left(-A(E_t^* - V_t^*) + B\right) - \frac{\beta}{\delta\kappa_v}$$

and hence

$$d\mathbb{E}[E_t^* - V_t^*] = \left(-(a + b\lambda + \frac{2}{\bar{\kappa}} A)\mathbb{E}[E_t^* - V_t^*] + \frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta \kappa_v} \right) dt.$$

This ODE has a unique solution which is

$$\mathbb{E}[E_t^* - V_t^*] = e^{-(a+b\lambda + \frac{2}{\bar{\kappa}} A)t} \left(q - p - \frac{\frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta\kappa_v}}{a + b\lambda + \frac{2}{\bar{\kappa}} A} \right) + \frac{\frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta\kappa_v}}{a + b\lambda + \frac{2}{\bar{\kappa}} A}.$$

Therefore, we have

$$\lim_{t\to\infty} \mathbb{E}[E_t^* - V_t^*] = \frac{\frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta\kappa_v}}{a + b\lambda + \frac{2}{\bar{\kappa}} A} = \frac{1}{a + b\lambda + \frac{2}{\bar{\kappa}} A} G_{max},$$

where the convergence takes place with an exponential rate. □

Proof of Proposition 7. As usual, we drop the index i for the sake of simplicity.

(i) According to Propositions 5 and 2, we have

$$G_{max} = \frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta\kappa_v} = \beta \left(\frac{2}{\bar{\kappa}} \frac{1 + \frac{A}{\delta\kappa_v}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A} - \frac{1}{\delta\kappa_v} \right).$$

Noticing that $\frac{2}{\bar{\kappa}} B - \frac{\beta}{\delta\kappa_v} > 0$ when condition (9) is satisfied, as $\beta > 0$, we deduce that

$$\frac{2}{\bar{\kappa}} \frac{1 + \frac{A}{\delta\kappa_v}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A} - \frac{1}{\delta\kappa_v} > 0$$

under this condition. It means that G_{max} increases linearly in β when condition (9) is satisfied.

(ii) Using the expression of A in Proposition 2, we get that

$$\frac{\partial G_{max}}{\partial \alpha} = \beta \frac{\frac{1}{\delta\kappa_v} (\delta + a + \lambda b) - \frac{2}{\bar{\kappa}}}{\left(\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A\right)^2} \frac{1}{\sqrt{1 + \frac{16}{\bar{\kappa}} \frac{T}{R^2}}} \frac{4}{\bar{\kappa}} \frac{1}{R} \frac{\lambda^i b^2}{\delta + \rho} \frac{2}{1 + b}.$$

Now,

$$\frac{1}{\delta\kappa_v} (\delta + a + \lambda b) - \frac{2}{\bar{\kappa}} < 0 \iff \frac{\kappa_v}{\kappa_c} > \frac{a + b\lambda}{\delta}.$$

Hence, as all the other terms are positive, G_{max} decreases with α when condition (9) is verified. Moreover, as T increases with α and A increases with α through T, we can see that $\frac{\partial G_{max}}{\partial \alpha}$ increases with α when condition (9) is verified, as it is negative under this condition. Hence, G_{max} is convex in α under condition (9). □

Proof of Proposition 8. As usual, we drop the index i for the sake of simplicity.

(i) According to Proposition 2, the constant in the optimal greening strategy is equal to

$$\frac{1}{\kappa_v}\left(\frac{\beta}{\delta} - B\right) = \beta\frac{1}{\kappa_v}\left(\frac{1}{\delta} - \frac{1+\frac{A}{\delta\kappa_v}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A}\right)$$

Now, using that $\frac{2}{\bar{\kappa}} = \frac{1}{\kappa_v} + \frac{1}{\kappa_c}$, we have

$$\frac{1}{\delta} > \frac{1+\frac{A}{\delta\kappa_v}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A} \iff 1 > \frac{\delta + \frac{A}{\kappa_v}}{\delta + a + \lambda b + \left(\frac{1}{\kappa_v} + \frac{1}{\kappa_c}\right)A}$$

which is always true as all parameters are positive. Hence, $\frac{1}{\delta} - \frac{1+\frac{A}{\delta\kappa_v}}{\delta+a+\lambda b+\frac{2}{\bar{\kappa}}A} > 0$, which means that $\frac{1}{\kappa_v}\left(\frac{\beta}{\delta} - B\right)$ increases linearly in β.

(ii) Using the expression of A in Proposition 2, we get that

$$\frac{\partial\left(\frac{1}{\kappa_v}\left(\frac{\beta}{\delta} - B\right)\right)}{\partial \alpha} = -\frac{1}{\kappa_v}\beta\frac{\frac{1}{\delta\kappa_v}(\delta + a + \lambda b) - \frac{2}{\bar{\kappa}}}{\left(\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A\right)^2}\frac{1}{\sqrt{1+\frac{16}{\bar{\kappa}}\frac{T}{R^2}}}\frac{8}{\bar{\kappa}^2}\frac{1}{R}\frac{\lambda^i b^2}{\delta + \rho}\frac{2}{1+b}.$$

which is positive under condition (9), according to the proof of Proposition 7 (ii). Moreover, using similar arguments as in that proof, we get that, under condition (9), $\frac{1}{\kappa_v}\left(\frac{\beta}{\delta} - B\right)$ is concave in α. □

3.4 Limiting cases

Proof of Proposition 9. Referring to Proposition 2 describing optimal controls in the general case, notice that $A^i = 0$ when $\alpha = 0$, and hence $B^i = \frac{\beta}{\delta+a+b\lambda^i}$. Therefore, optimal controls are given by (2.1). □

Proof of Proposition 10. We drop the exponent i in the proof for simplicity. According to Proposition 9, $c_t^* = \frac{1}{\kappa_c}\frac{\beta}{\delta+a+b\lambda}$. As we have assumed that $\beta > 0$ in this subsection, we always have $c_t^* > 0$. Moreover, using the expressions of c_t^*, v_t^* given in Proposition 9, we have

$$c_t^* > v_t^* \iff \frac{v_t^*}{c_t^*} < 1 \iff \frac{\kappa_v^i}{\kappa_c^i} > \frac{a + b\lambda^i}{\delta}. \tag{3.27}$$

(i) Let us assume that condition (2.2) is satisfied. Referring to Definition 1, it implies that the company greenwashes if, and only if, $E_t^* \geq V_t^*$. Now, if the company is overrated at time t, i.e. if $E_t^* \geq V_t^*$, as $0 \leq a, b \leq 1$ and $c_t^* > 0, c_t^* > v_t^*$, the only possibility to get $E_s^* < V_s^*$ is through the measurement noise zdW_t in (2a). Indeed, referring to the explicit expression of $X_t^* = E_t^* - V_t^*$ in (3.24), all terms are positive except the Itô integral which can be negative, and which represents the measurement error.

Then, greenwashing effort can be computed as follows, using Proposition 9:

$$G^\beta := c_t^* - v_t^* = \frac{1}{\kappa_c}\frac{\beta}{\delta + a + b\lambda} - \frac{1}{\kappa_v}\left(\frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda}\right) = \frac{2}{\bar{\kappa}}\frac{\beta}{\delta + a + b\lambda} - \frac{\beta}{\delta\kappa_v}.$$

Moreover, $G^\beta > 0$ as $c_t^* - v_t^* > 0$ under condition (2.2).

(ii) If condition (2.2) is not verified, we have $c_t^* \leq v_t^*$ for all t, and hence the company never greenwashes. □

Proof of Proposition 11. We drop the exponent i for simplicity. Inserting optimal strategies of equation (2.1) into the dynamics of the environmental score (2a), one can deduce that $\mathbb{E}[E_t^* - V_t^*]$ verifies the following ODE:

$$d\mathbb{E}[E_t^* - V_t^*] = \left(-(a + \lambda b)\mathbb{E}[E_t^* - V_t^*] + \frac{2}{\bar{\kappa}}\frac{\beta}{\delta + a + b\lambda^i} - \frac{\beta}{\delta\kappa_v}\right)dt,$$
$$\mathbb{E}[E_0^* - V_0^*] = q - p.$$

The solution to this ODE exists, is unique and given by

$$\mathbb{E}[E_t^* - V_t^*] = e^{-(a+b\lambda)t}(q - p) + \left(\frac{2}{\bar{\kappa}}\frac{\beta}{\delta + a + b\lambda^i} - \frac{\beta}{\delta\kappa_v}\right)\frac{1}{a + b\lambda}\left(1 - e^{-(a+b\lambda)t}\right).$$

Therefore,

$$\mathbb{E}[E_t^* - V_t^*] \xrightarrow{t\to\infty} \frac{1}{a+b\lambda}\left(\frac{2}{\bar\kappa}\frac{\beta}{\delta+a+b\lambda} - \frac{\beta}{\delta\kappa_v}\right) =: L_\beta.$$

(Reminding that $a+b\lambda>0$ as assumed after Definition 3.) Now,

$$L_\beta > 0 \iff \frac{2}{\bar\kappa}\frac{\beta}{\delta+a+b\lambda} - \frac{\beta}{\delta\kappa_v} > 0 \iff \frac{\kappa_v}{\kappa_c} > \frac{a+b\lambda}{\delta}.$$

This concludes the proof. □

Proof of Proposition 12. Referring to Proposition 2, when $\beta=0$, we have that $B^i=0$, and A^i is unchanged. This gives optimal controls as in (2.3).

(i) and (ii) can be deduced from the shapes of the optimal controls in equation (2.3), using that $A^i>0$ if $\alpha>0$, as it can be seen in equation (7), and recording the definitions of the two types of environmental communication (green and brown communications). □

Proof of Proposition 13. According to equation (2.3), as $A^i \geq 0$ (refer to Proposition 2), when $E_t^{i,*} \geq V_t^{i,*}$, $c_t^* \leq 0$. And, when $E_t^{i,*} < V_t^{i,*}$, $c_t^{i,*} > 0$. Therefore, the cases $E_t^* \geq V_t^*$ and $c_t^* > 0$ never happen at the same instants. Referring to the definition of greenwashing (Definition 1), one can conclude that the company never practices greenwashing in this limiting case.

Now, let us show that $\lim_{t\to\infty}\mathbb{E}[E_t^{i,*} - V_t^{i,*}] = 0$. Inserting optimal strategies of equation (2.3) into the dynamics of the environmental score (2a), one can deduce that $\mathbb{E}[E_t^* - V_t^*]$ verifies the following ODE:

$$\begin{aligned} &d\mathbb{E}[E_t^* - V_t^*] = -(a+\lambda b+\frac{2}{\bar\kappa}A)\mathbb{E}[E_t^* - V_t^*]dt, \\ &\mathbb{E}[E_0^* - V_0^*] = q-p. \end{aligned}$$

The solution to this ODE exists, is unique and given by

$$\mathbb{E}[E_t^* - V_t^*] = e^{-(a+b\lambda+\frac{2}{\bar\kappa}A)t}\,(q-p)\,.$$

Therefore, $\mathbb{E}[E_t^* - V_t^*] \xrightarrow{t\to\infty} 0$, where the convergence takes place with an exponential rate. $\square$

3.5 Empirics

Lemma 3.9. *The bias of the Within estimate under weak exogeneity tends towards zero at a rate faster than or equal to 1/T.*

Proof of Lemma 3.9. Let us prove this lemma by considering the following generic specification, for all $i \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, T\}$:

$$Y_{i,t} = \alpha_i + X_{i,t}\beta + \varepsilon_{i,t},$$

where for each t, $(X_{i,t}, \varepsilon_{i,t})_i$ are integrable i.i.d. variables, for each i, $(X_{i,t}, \varepsilon_{i,t})$ is stationary, and $\forall t' \geq t$, $\mathbb{E}(X_{i,t}\varepsilon_{i,t'}) = 0$ (weak exogeneity assumption).

Let us set $\overline{X}_i = \frac{1}{T}\sum_{t=1}^T X_{i,t}$ and $\tilde{X}_{i,t} = X_{i,t} - \overline{X}_i$, and define $\bar{Y}_i$, $\tilde{Y}_{i,t}$, $\bar{\varepsilon}_i$, $\tilde{\varepsilon}_{i,t}$ similarly. The Within estimator, $\hat{\beta}$, verifies

$$\hat{\beta} = \left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T(\tilde{X}_{i,t})^2\right)^{-1}\left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\tilde{X}_{i,t}\tilde{Y}_{i,t}\right),$$

that is,

$$\hat{\beta} = \beta + \left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T(\tilde{X}_{i,t})^2\right)^{-1}\left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\tilde{X}_{i,t}\varepsilon_{i,t}\right),$$

where $\tilde{\varepsilon}_{i,t}$ is replaced by $\varepsilon_{i,t}$ because $\frac{1}{T}\sum_{t=1}^T\tilde{X}_{i,t}\bar{\varepsilon}_i = \bar{\varepsilon}_i\frac{1}{T}\sum_{t=1}^T\tilde{X}_{i,t} = 0$. Therefore, we can write the bias of the Within estimation as

$$\hat{\beta} - \beta = \left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T(\tilde{X}_{i,t})^2\right)^{-1}\left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\tilde{X}_{i,t}\varepsilon_{i,t}\right).$$

By the law of large numbers, writing (X_t, ε_t) with the same distribution as $(X_{i,t}, \varepsilon_{i,t})$ for any i,

$$\operatorname*{plim}_{N\to\infty}\left(\hat{\beta}-\beta\right) = \left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\tilde{X}_t^2\right]\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\tilde{X}_t\varepsilon_t\right]\right),$$

writing plim for the convergence in probability.

Now, for each t, $\mathbb{E}\left(\tilde{X}_t\varepsilon_t\right) = \mathbb{E}\left((X_t-\overline{X})\varepsilon_t\right) = -\mathbb{E}\left(\overline{X}\varepsilon_t\right)$. Therefore, because $(X_t, \varepsilon_t)_t$ is stationary and $\mathbb{E}[X_t\varepsilon_t] = 0$ (from the weak exogeneity assumption), one can rewrite this bias as

$$\operatorname*{plim}_{N\to\infty}\left(\hat{\beta}-\beta\right) = \left(\mathbb{E}\left[\tilde{X}_t^2\right]\right)^{-1}\left(-\mathbb{E}\left(\overline{X}\bar{\varepsilon}\right)\right),\ t\in\{1,\ldots,T\}.$$

However, from Cauchy-Schwartz,

$$|\mathbb{E}\left(\overline{X}\bar{\varepsilon}\right)| \le \left(Var(\overline{X})Var(\bar{\varepsilon})\right)^{1/2} = O(1/T)$$

if X_t and ε_t are weakly dependent.

Therefore, the bias tends to zero, and its limit in probability is upper-bounded by a variable that tends to zero at a rate $1/T$, which proves Lemma 3.9.

□

4 Calibration

The reference calibration used in Section 3 is made to illustrate the properties of the model for a generic company.

We calibrate the frequency of controversies, λ, using the Environmental Controversy score provided by Covalence: λ is the average frequency for which this score is above 25 (over 100) across the 13,298 companies in the whole Covalence database from January 2009 to December 2022. We assume that when a controversy occurs, the fundamental environmental value of the company is fully revealed ($b = 1$). We also assume that the rating agency progressively recovers the fundamental environmental value of the company over two years on average

($a = 0.5$). We choose the marginal unit cost of environmental communication relative to the marginal unit cost of greening ($\kappa_v/\kappa_c = 50$) in line with the ratio of a EUR 3,000,000 green bond emission to its certification costs (of the order of EUR 60,000).

We set the pro-environmental sensitivity of the investor, β, equal to the generic value of 1. Since the green premium and the misrating penalty premium are homogeneous metrics, we also assume that $\alpha = 1$. It is worth noting that α and β do not impact the "ON-OFF" greenwashing condition (equation (9)), but only contribute to scaling greening, communication, and greenwashing efforts. We consider a rate of time preference of 10% for both the company and the investor.

As such, the calibration verifies the following two realistic conditions:

1. It is much more costly to abate than to do environmental communication ($\kappa_v >> \kappa_c$).

2. The relative marginal unit costs κ_v/κ_c, imperfection of information $a + b\lambda$, and rate of time preference δ, are so that condition (2.2) is satisfied.

In short, the calibration is reported in Table 4.1.

Table 4.1: Calibration.

Parameter	Value
a	0.5
b	1
λ	7.5%
κ_c	1
κ_v	50
β	1
α	1
ρ	0.1
δ	0.1

5 Extension with interaction between companies

The n-player game In the new program, the investor normalizes each company's environmental rating by the average environmental score among the n companies. The investor's extended program is set as follows:

$$\sup_{\omega \in \mathbb{A}^\omega} \mathbb{E}\bigg[\int_0^\infty e^{-rt}\Big\{\omega_t' dS_t - \frac{\gamma}{2}\langle \omega' dS\rangle_t + \omega_t'\big(\beta \frac{E_t}{h(\frac{1}{n}\sum_i E_t^i)} - \alpha M_t\big)dt\Big\}\bigg],$$

with h a regular function inferiorly bounded by a strictly positive constant and approximating the identity function on $\mathbb{R}_+$. Note that, when h is the constant function equal to 1, this program is the same as in Section 2.

Similarly to the initial problem, equilibrium expected returns can easily be deduced from this new program, as is done in the next Proposition.

Proposition 15 (Equilibrium expected returns in the n-player game). *Let us assume that E, M, solutions of dynamics* (5.29), *verify $E, M \in \mathbb{H}_n^2(\delta^I)$. Moreover, let us define S as a solution to (1) and the set of admissible strategies $\mathbb{A}^\omega$ for the program of the investor as $\mathbb{A}^\omega := \mathbb{H}_n^2(\delta^I)$.*

Then, the optimal portfolio choice of the investor is the pointwise solution

$$\omega_t^* = \frac{1}{\gamma}\Sigma^{-1}(\mu_t + \beta\frac{E_t}{h(\frac{1}{n}\sum_i E_t^i)} - \alpha M_t),$$

and equilibrium expected returns are

$$\mu_t = \gamma\Sigma\mathbf{1}_n - \beta\frac{E_t}{h(\frac{1}{n}\sum_i E_t^i)} + \alpha M_t.$$

Proof of Proposition 15. Under the assumptions of the proposition, the investor's program can be rewritten as

$$\sup_{\omega\in\mathbb{A}^\omega}\mathbb{E}\left[\int_0^\infty e^{-\delta^I t}\omega_t'\left(\mu_t+\beta\frac{E_t}{h(\frac{1}{n}\sum_i E_t^i)}-\alpha M_t-\frac{\gamma}{2}\Sigma\omega_t\right)dt\right]$$
$$=\sup_{\omega\in\mathbb{A}^\omega}\mathbb{E}\left[\int_0^\infty e^{-\delta^I t}\left\{-\frac{\gamma}{2}(\omega_t-\omega_t^*)'\Sigma(\omega_t-\omega_t^*)+\frac{\gamma}{2}\omega_t^{*\prime}\Sigma\omega_t^*\right\}dt\right].$$

The optimal portfolio choice of the investor is thus the pointwise solution ω_t^*. In addition, as the quantity of each asset is assumed to be normalised to one in the market, writing $\mathbf{1}_n$ a vector of ones of size n, market clearing condition writes:

$$\forall t,\ \omega_t^*=\mathbf{1}_n.$$

Equilibrium expected returns are therefore

$$\mu_t=\gamma\Sigma\mathbf{1}_n-\beta\frac{E_t}{h(\frac{1}{n}\sum_i E_t^i)}+\alpha M_t.$$

□

Plugging these new equilibrium expected returns in each company's program, the program of company i becomes the following:

$$\inf_{(v^i,c^i)\in\mathbb{A}}\mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\Gamma_i-\beta\frac{E_t^i}{h(\frac{1}{n}\sum_i E_t^i)}+\alpha M_t^i+\frac{\kappa_v}{2}(v_t^i)^2+\frac{\kappa_c}{2}(c_t^i)^2\right)dt\right]. \qquad (5.28)$$

Companies' programs are now interacting, as the above objective depends on the average environmental rating among companies. Moreover, they are no more linear quadratic: each company controls both the numerator and the denominator in the environmental score term of its cost of capital, $E_t^i/h(\frac{1}{n}\sum_i E_t^i)$. As a result, the n-player game cannot be solved in explicit form. To approximate the Nash equilibria of this game with interpretable quantities, we formulate and solve the mean field limit of this game.

A Greenwashing mean field game In order to define a mean field game (MFG) which approximates the Greenwashing n-player game, we need to make two additional assumptions. (i) Companies are homogeneous: all parameters are the same for each company. (ii) Their environmental scores are driven by idiosyncratic noises: $(W^i, \widetilde{N}^i, \widehat{N}^i)_i$ are assumed to be independent and identically distributed (i.i.d). Hence, at the mean field limit ($n \to \infty$), we work with a representative company which admits $(E, V, M) \in \mathbb{R}^3$ as a state variable, solution to the following dynamics:

$$\begin{cases} dE_t = a(V_t - E_t)dt + (V_{t-} - E_{t-})d\widetilde{N}_t + c_t dt + z dW_t, \quad E_0 = \tilde{q}, \\ dV_t = v_t dt, \quad V_0 = \tilde{p}, \\ dM_t = -\rho M_t dt + (V_{t-} - E_{t-})^2 d\widehat{N}_t, \quad M_0 = \tilde{u}, \end{cases} \tag{5.29}$$

with W a one-dimensional Brownian motion, $\widetilde{N}$ is a compound Poisson process with intensity $\lambda \in \mathbb{R}_+^*$ and jump size distribution $B(1, 1/b - 1)$, independent from W, $\widehat{N}$ is a compound Poisson process such that $\Delta\widehat{N}_t = (\Delta\widetilde{N}_t)^2$ for all $t \geq 0$, and $(\tilde{q}, \tilde{p}, \tilde{u})$ is a square integrable random variable valued in $\mathbb{R}^2 \times \mathbb{R}_+$ and independent from the triple $(W, \widetilde{N}, \widehat{N})$. From now on, for the n-player game, we keep the exponent i to index companies, while the state variables of the representative company considered at the mean field limit is distinguished by the absence of exponent.[31] Note that, under the assumptions (i) and (ii), the environmental score, the environmental value, and the misrating score of the representative company and of the n companies in the n-player game follow the same distribution.

At the mean field limit ($n \to \infty$), by the law of large numbers, we expect the average environmental score of companies, $\lim_{n\to\infty} \frac{1}{n}\sum_i E_t^i$, to be a deterministic function, $m \in \mathcal{C}^1([0,T], \mathbb{R})$. Hence, the program of the representative company at the mean field limit

[31] Hence, the absence of exponent no longer identifies an n-dimensional vector, but a one-dimensional variable characterizing the representative company in the mean field version of the Greenwashing game.

is equivalent to the following, up to a constant:

$$\sup_{(v,c)\in\mathbb{A}} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\left(\beta\frac{E_t}{h(m_t)} - \alpha M_t - \frac{\kappa_v}{2}(v_t)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right],$$

with (E, V, M) solution of equation (5.29).

This program has an infinite horizon, while it has now a time-dependent parameter, $1/h(m_t)$, which makes it not very well suited for infinite horizon resolution. Hence, we approximate its solution by a finite horizon equivalent, with a horizon $T \in \mathbb{R}_+$ big enough:

$$\sup_{(v,c)\in\mathbb{A}_T} \mathbb{E}\left[\int_0^T e^{-\delta t}\left(\beta\frac{E_t}{h(m_t)} - \alpha M_t^i - \frac{\kappa_v}{2}(v_t)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right], \tag{5.30}$$

with a new set of admissible strategies, $\mathbb{A}_T$, which is the set of $\mathbb{F}$-progressively measurable $\mathbb{R}^2$-valued processes which verify $\mathbb{E}\left[\int_0^T |v_t|^2 + |c_t|^2 dt\right] < \infty$. This program, associated with the state variable dynamics of the representative company described in equation (5.29), characterizes the Greenwashing Mean Field Game (MFG) that we solve in this extension. Before solving it, we need to define the notion of solution to a mean field game, that we call a mean field equilibrium (MFE), and which is the equivalent of a Nash equilibrium at the mean field limit.

Definition 6 (Mean field equilibrium of the Greenwashing MFG)**.** Consider the functional

$$J(v, c, m) := \mathbb{E}\left[\int_0^T e^{-\delta t}\left(\beta\frac{E_t}{h(m_t)} - \alpha M_t - \frac{\kappa_v}{2}(v_t)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right], \tag{5.31}$$

defined for any admissible strategy $(v, c) \in \mathbb{A}_T$ and deterministic function of time $m \in \mathcal{C}^1([0, T], \mathbb{R})$, with (E, V, M) solution to equation (5.29) when the environmental strategy (v, c) is employed. Then, the triplet $(v^*, c^*, m^*) \in \mathbb{A}_T \times \mathcal{C}^1([0, T], \mathbb{R})$ is a mean field equilibrium of the Greenwashing MFG if, and only if,

(i) $\forall (v, c) \in \mathbb{A}_T, \quad J(v^*, c^*, m^*) \geq J(v, c, m^*)$,

(ii) $\forall t \in [0, T], \quad m_t^* = \mathbb{E}[E_t^*]$, with (E^*, V^*, M^*) solution to equation (5.29) when the strategy (v^*, c^*) is employed.

A mean field equilibrium is so that the representative company adopts an optimal strategy for a given environmental score average, and that this environmental score average represents the average environmental score of companies acting optimally. As each company is represented by the representative company, it must represent the expected environmental score of the representative company acting optimally. Hence, to identify a mean field equilibrium, one first needs to identify the "best response" of the representative company to a given environmental score average, and then to identify the fixed point(s) of the resulting best response functional.

Optimal strategy for a given environmental score average For a given environmental score average, written $m \in \mathcal{C}^1([0,T], \mathbb{R})$, the optimal communication and greening strategy of the representative company can be computed explicitly.

Proposition 16 (Optimal strategy in the Greenwashing MFG)**.** *For a given environmental score average, $m \in \mathcal{C}^1([0,T], \mathbb{R})$, the optimal environmental communication effort, $\hat{c}$, and greening effort, $\hat{v}$, of the representative company are as follows:*

$$\hat{c}_t = \frac{1}{\kappa_c}\left(B(t) + A(t)(\hat{E}_t - \hat{V}_t)\right), \qquad \hat{v}_t = \frac{1}{\kappa_v}\left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds - B(t) - A(t)(\hat{E}_t - \hat{V}_t)\right),$$

where

$$B(t) = \beta \int_t^T e^{\int_t^s \left(\frac{2}{\bar{\kappa}} A(u) - \delta - a - \lambda b\right) du} \left(\frac{1}{h(m_s)} - \frac{A(s)}{\kappa_v} \int_s^T \frac{e^{-\delta(u-s)}}{h(m_u)} du\right) ds,$$

and A is the unique solution, negative, to the Riccati equation

$$\dot{A}(t) + \frac{2}{\bar{\kappa}} A(t)^2 - \left(\delta + 2a + \frac{2\lambda b}{b+1}\right) A(t) + \frac{4\lambda b^2}{b+1}\left(\frac{\alpha}{\delta+\rho} e^{-(\delta+\rho)(T-t)} - \frac{\alpha}{\delta+\rho}\right) = 0, \quad A(T) = 0, \tag{5.32}$$

and where $\hat{E}, \hat{V}$ are solution to the dynamics (5.29) *when the optimal strategy $(\hat{v}, \hat{c})$ is employed.*

Proof of Proposition 16. Let us define the value function, $\hat{w}$, of the representative company:

$$\hat{w}(q,p,u) := \sup_{(v^i,c^i)\in\mathbb{A}_T} \mathbb{E}\left[\int_0^T e^{-\delta t}\left(\beta\frac{E_t^q}{h(m_t)} - \alpha M_t^u + \frac{\kappa_v}{2}(v_t)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right]$$

with the following constraints. The state variables of the representative company's program are the tridimensional process (E^q, V^p, M^u) which is the unique strong solution (Protter, 2005, Chapter 5, Theorem 52) to the following SDEs:

$$\begin{cases} dE_t^q = a(V_t^p - E_t^q)dt + (V_{t-}^p - E_{t-}^q)d\widetilde{N}_t + c_t dt + z dW_t, \quad E_0^q = q, \\ dV_t^p = v_t dt, \quad V_0^p = p, \\ dM_t^u = -\rho M_t^u dt + (V_{t-}^p - E_{t-}^q)^2 d\widehat{N}_t, \quad M_0^u = u, \end{cases} \tag{5.33}$$

for $(q,p,u) \in \mathcal{Y}$, $\mathcal{Y} := \mathbb{R}^2 \times \mathbb{R}_+$ and $(v,c) \in \mathbb{A}_T$.

The program can be rewritten, up to a constant (depending on m), as follows:

$$\sup_{(v,c)\in\mathbb{A}_T} \mathbb{E}\left[\int_0^T e^{-\delta t}\left(\frac{\beta}{h(m_t)}(E_t - V_t) - \alpha M_t - \frac{\kappa_v}{2}\left(v_t - \frac{\beta}{\kappa_v}\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2 - \frac{\kappa_c}{2}(c_t)^2\right)dt\right].$$

Then, remark that

$$\begin{aligned} &\frac{\kappa_v}{2}\left(v_t - \frac{\beta}{\kappa_v}\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2 + \frac{\kappa_c}{2}(c_t)^2 \\ &\quad = \frac{\bar{\kappa}}{4}\left(c_t - v_t + \frac{\beta}{\kappa_v}\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2 + \frac{1}{2(\kappa_v+\kappa_c)}\left(\kappa_c c_t + \kappa_v v_t - \beta\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2. \end{aligned}$$

Let $\xi_t = c_t - v_t$ with $(v,c) \in \mathbb{A}_T$ and introduce the new state process $X_t = E_t^q - V_t^p$, so that

$$dX_t^x = -aX_t^x dt - X_{t-}^x d\widetilde{N}_t + \xi_t dt + z dW_t, \quad X_0 = x = q - p,$$

$$dM_t^u = -\rho M_t^u dt + (X_{t-}^x)^2 d\widehat{N}_t, \quad M_0 = u.$$

We have $\hat{w}(q,p,u) = \tilde{w}(x,u)$, with

$$\tilde{w}(x,u) = \sup_{\substack{\xi=c-v,\\(v,c)\in\mathbb{A}_T}} \mathbb{E}\Bigg[\int_0^T e^{-\delta t}\Bigg(\frac{\beta}{h(m_t)}X_t^x - \alpha M_t^u - \frac{\bar{\kappa}}{4}\left(c_t - v_t + \frac{\beta}{\kappa_v}\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2 - \frac{1}{2(\kappa_v+\kappa_c)}\left(\kappa_c c_t + \kappa_v v_t - \beta\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2\Bigg)dt\Bigg].$$

It is then clear that at optimum, the controls satisfy

$$\kappa_c c_t + \kappa_v v_t - \beta\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds = 0,$$

and that we can concentrate on the following auxiliary two dimensional problem:

$$\tilde{w}(x,u) = \sup_{\substack{\xi=c-v,\\(v,c)\in\mathbb{A}_T}} \mathbb{E}\left[\int_0^T e^{-\delta t}\left(\frac{\beta}{h(m_t)}X_t^x - \alpha M_t^u - \frac{\bar{\kappa}}{4}\left(\xi_t + \frac{\beta}{\kappa_v}\int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)}ds\right)^2\right)dt\right].$$

Let $\mathcal{X}_T := [0,T]\times\mathbb{R}\times\mathbb{R}_+$, and $(t,x,u)\in\mathcal{X}_T$. We define, on $\mathcal{X}_T$, the value function in time as follows:

$$w(t,x,u) = \sup_{\xi\in\mathbb{A}_T^\xi} \mathbb{E}\left[\int_t^T e^{-\delta(s-t)} f_T(s, X_s^{t,x}, M_s^{t,u})ds\right],$$

with $f_T(s,x,u,\xi) := \frac{\beta}{h(m_s)}x - \alpha u - \frac{\bar{\kappa}}{4}\left(\xi + \frac{\beta}{\kappa_v}\int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)}dr\right)^2$, $\mathbb{A}_T^\xi$ the set of $\mathbb{F}$-progressively measurable $\mathbb{R}$-valued processes ξ which verify $\mathbb{E}\left[\int_0^T |\xi_t|^2 dt\right] < \infty$, and the auxiliary bidimensional state variables process $(X^{t,x}, M^{t,u})$ as the unique strong solution to the following SDEs (Protter, 2005, Chapter 5, Theorem 52):

$$\begin{cases} dX_s^{t,x} = -aX_s^{t,x}ds - X_{s-}^{t,x}d\widetilde{N}_s + \xi_s ds + z dW_s, \quad X_t = x, \\ dM_s^{t,u} = -\rho M_s^{t,u}ds + (X_{s-}^{t,u})^2 d\widehat{N}_s, \quad M_t = u. \end{cases} \tag{5.34}$$

Moreover, note that for any $\xi\in\mathbb{A}_T^\xi$, the bidimensional auxiliary state variable (5.34) admits the following explicit solution:

$$X_s^{t,x} = \mathcal{E}_s^t x + \mathcal{E}_s^t \int_t^s (\mathcal{E}_r^t)^{-1} \{\xi_r dr + z dW_r\}, \tag{5.35}$$

$$M_s^{t,u} = e^{-\rho(s-t)} u + \int_t^s e^{-\rho(s-r)} (X_{r-}^{t,x})^2 d\widehat{N}_r, \tag{5.36}$$

with $\mathcal{E}_s^t = e^{-at} \prod_{s \leq r \leq t} (1 - \Delta \widetilde{N}_r)$, and writing $0^0 = 1$.

HJB equation For $b < 1$, the value function satisfies the following HJB equation, for all $(t, x, u) \in \mathcal{X}_T$, omitting the argument (t, x, u) of the function w and its partial derivatives when it is clear:

$$\max_{\xi \in \mathbb{R}} \Bigg\{ \frac{\beta}{h(m_t)} x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\kappa_v} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 - \delta w + \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}(-ax + \xi)$$
$$- \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda(1/b - 1) \int_0^1 (1-y)^{1/b-2} \left[w(x(1-y), u + y^2 x^2) - w(x, u) \right] dy \Bigg\} = 0, \tag{5.37}$$

or in other words, replacing ξ by the optimizing function $\xi^*(t, x, u) := \frac{2}{\bar{\kappa}} \frac{\partial w}{\partial x} - \frac{\beta}{\kappa_v} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds$,

$$\frac{\beta}{h(m_t)} x - \alpha u + \frac{1}{\bar{\kappa}} \left(\frac{\partial w}{\partial x} \right)^2 - \delta w + \frac{\partial w}{\partial t} - \frac{\partial w}{\partial x} \left(ax + \frac{\beta}{\kappa_v} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)$$
$$- \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda(1/b - 1) \int_0^1 (1-y)^{1/b-2} \left[w(x(1-y), u + y^2 x^2) - w(x, u) \right] dy = 0.$$

Let us use the ansatz

$$\phi(t, x, u) = \frac{1}{2} A(t) x^2 + B(t) x + C(t) u + w_0(t).$$

Substituting this into the equation and collecting terms with the same powers of u and x, we get:

$$-\alpha + \dot{C}(t) - (\rho + \delta) C(t) = 0, \quad C(T) = 0$$
$$\dot{A}(t) + \frac{2}{\bar{\kappa}} A(t)^2 - \left(\delta + 2a + \frac{2\lambda b}{b+1} \right) A(t) + \frac{4\lambda b^2}{b+1} C(t) = 0, \quad A(T) = 0$$
$$\dot{B}(t) + \left(\frac{2}{\bar{\kappa}} A(t) - \delta - a - \lambda b \right) B(t) + \frac{\beta}{h(m_t)} - A(t) \frac{\beta}{\kappa_v} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds = 0, \quad B(T) = 0,$$

and the optimal control is, for all $s \in [t, T]$,

$$\hat{\xi}_s = \frac{2}{\bar{\kappa}} \left(A(s)\hat{X}_s^{t,x} + B(s) \right) - \frac{\beta}{\kappa_v} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr,$$

with $\hat{X}^{t,x}$ solution of the dynamics (5.34) when the strategy $\hat{\xi}$ is employed.

Solutions to the above solvable equations are :

$$C(t) = \frac{\alpha}{\delta + \rho} e^{-(\delta+\rho)(T-t)} - \frac{\alpha}{\delta + \rho},$$

$$B(t) = \beta \int_t^T e^{\int_t^s \left(\frac{2}{\bar{\kappa}} A(u) - \delta - a - \lambda b \right) du} \left(\frac{1}{h(m_s)} - \frac{A(s)}{\kappa_v} \int_s^T \frac{e^{-\delta(u-s)}}{h(m_u)} du \right) ds.$$

Existence and negativity of the Riccati solution Before stating the verification argument, let us show that the Riccati equation (5.32) admits a unique solution, which is negative. Let the Riccati equation (5.32) be rewritten, with a suitable function f, as $\dot{A}(t) = f(t, A(t))$. Fix a large constant M. The equation $\dot{A}_M(t) = -M \vee f(t, A_M(t)) \wedge M$ has a unique C^1 solution on $[0, T]$ by Cauchy-Lipschitz theorem. Let us define A^-, A^+ solutions of the following two Riccati equations:

$$\dot{A}^-(t) = -\frac{2}{\bar{\kappa}} A^-(t)^2 + (\delta + 2a + \frac{2\lambda b}{b+1}) A^-(t) + \frac{4\lambda b^2}{b+1} \frac{\alpha}{\delta + \rho}, \qquad A^-(T) = 0, \tag{5.38}$$

$$\dot{A}^+(t) = -\frac{2}{\bar{\kappa}} A^+(t)^2 + (\delta + 2a + \frac{2\lambda b}{b+1}) A^+(t), \qquad A^+(T) = 0. \tag{5.39}$$

They admit the following explicit solutions: for all $t \in [0, T]$, $A^+(t) = 0$, and

$$A^-(t) = \frac{4\lambda b^2 \alpha}{(\delta + \rho)(1 + b)} \frac{e^{2\sqrt{R}(T-t)} - 1}{-\left(\sqrt{R} + (a + \frac{\delta}{2} + \frac{\lambda b}{b+1}) \right) e^{2\sqrt{R}(T-t)} + a + \frac{\delta}{2} + \frac{\lambda b}{b+1} - \sqrt{R}},$$

$$R = \left(a + \frac{\delta}{2} + \frac{\lambda b}{b+1} \right)^2 + \frac{4\lambda b^2}{1+b} \frac{2}{\bar{\kappa}} \frac{\alpha}{\delta + \rho}.$$

Let us rewrite them $\dot{A}^-(t) = g(t, A^-(t))$, $\dot{A}^+(t) = h(t, A^+(t))$ by defining g, h accordingly. We have that, $\forall t \in [0, T)$, $\forall x \in \mathbb{R}$, $\quad h(t, x) \leq f(t, x) \leq g(t, x)$, and $A^+(T) = A(T) = A^-(T) = 0$. Moreover, note that A^- and A^+ are bounded according to their explicit solutions. Then, by the comparison theorem we have that for M sufficiently large, $\forall t \in [0,T)$, $A^+(t) \geq A_M(t) \geq A^-(t)$. By the boundedness of A^- and A^+, we have that, for M sufficiently large, $f(t, A_M(t)) \in [-M, M]$, which means that A_M solves the original Riccati equation on $[0,T]$. Finally, Cauchy-Lipschitz theorem guarantees that $A_M = A$ is unique. In particular, as $A \leq A^+ = 0$, A is negative.

Verification argument for the auxiliary program Let us define on $\mathcal{X}_T$ the function

$$\phi(t,x,u) = \frac{1}{2}A(t)x^2 + B(t)x + C(t)u + w_0(t).$$

Let us show that $w = \phi$.

(i) Let $\xi \in \mathbb{A}_T^{\xi}$ and $(t,x,u) \in \mathcal{X}_T$. Let $s \in [t,T]$. By Itô's formula, for the stopping time τ_n defined below, we have:

$$e^{-\delta(s\wedge\tau_n)}\phi(s\wedge\tau_n, X^{t,x}_{s\wedge\tau_n}, M^{t,u}_{s\wedge\tau_n}) = e^{-\delta t}\phi(t,x,u) + \int_t^{s\wedge\tau_n} e^{-\delta r}\left(-\delta\phi + \frac{\partial\phi}{\partial t} + \mathcal{L}^{\xi_r}\phi\right)(r, X_r^{t,x}, M_r^{t,u})dr + \int_t^{s\wedge\tau_n} e^{-\delta r}\frac{\partial\phi}{\partial x}(r, X_r^{t,x}, M_r^{t,u})z dW_r,$$

with the stopping time

$$\tau_n := \inf\{s \geq t : \int_t^s e^{-\delta r}|\frac{\partial\phi}{\partial x}(r, X_r^{t,x}, M_r^{t,u})|^2 dr \geq n\}, \quad \forall n \in \mathbb{N},$$

using the convention that $\inf\{\emptyset\} = \infty$, and the operator $\mathcal{L}^{\xi}\phi$ defined as follows, omitting the argument (t,x,u) of the function ϕ and its partial derivatives when it is clear:

$$\forall (t,x,u) \in \mathcal{X}_T, \quad \mathcal{L}^{\xi}\phi(t,x,u) := \frac{\partial\phi}{\partial x}(-ax+\xi) - \frac{\partial\phi}{\partial u}\rho u + \frac{z^2}{2}\frac{\partial^2\phi}{\partial x^2} + \lambda(1/b-1)\int_0^1 (1-y)^{1/b-2}\left[\phi(x(1-y), u+y^2x^2) - \phi(x,u)\right]dy.$$

The stopped stochastic integral is a martingale, and by taking the expectation we get

$$\mathbb{E}\left[e^{-\delta(s\wedge\tau_n)}\phi(s\wedge\tau_n, X^{t,x}_{s\wedge\tau_n}, M^{t,u}_{s\wedge\tau_n})\right] = \mathbb{E}\left[e^{-\delta t}\phi(t,x,u) + \int_t^{s\wedge\tau_n} e^{-\delta r}\left(-\delta\phi + \frac{\partial\phi}{\partial t} + \mathcal{L}^{\xi_r}\phi\right)(r, X^{t,x}_r, M^{t,u}_r)dr\right]$$
$$\leq e^{-\delta t}\phi(t,x,u) - \mathbb{E}[\int_t^{s\wedge\tau_n} e^{-\delta r} f_T(r, X^{t,x}_r, M^{t,u}_r, \xi_r)dr],$$

using equation (5.37), as ξ is any admissible control. By Lemmas 5.2 and 5.3, we may apply the dominated convergence theorem and send n to infinity:

$$\mathbb{E}[e^{-\delta s}\phi(s, X^{t,x}_s, M^{t,u}_s)] \leq e^{-\delta t}\phi(t,x,u) - \mathbb{E}[\int_t^{s} e^{-\delta r} f_T(r, X^{t,x}_r, M^{t,u}_r, \xi_r)dr]. \tag{5.40}$$

By sending now s to T, as ϕ is continuous and $\phi(T, X^{t,x}_T, M^{t,u}_T) = 0$, using again Lemmas 5.2 and 5.3, we then deduce

$$\phi(t,x,u) \geq \mathbb{E}\left[\int_t^T e^{-\delta(r-t)} f_T(r, X^{t,x}_r, M^{t,u}_r, \xi_r)dr\right], \quad \forall \xi \in \mathbb{A}^{\xi}_T,$$

and so $\phi \geq w$ on $\mathcal{X}_T$.

(ii) By repeating the above arguments and observing that the optimal control $\hat{\xi}$ achieves equality in (5.40) by construction, we have

$$\mathbb{E}[e^{-\delta s}\phi(s, X^{t,x}_s, M^{t,u}_s)] = e^{-\delta t}\phi(t,x,u) - \mathbb{E}[\int_t^{s} e^{-\delta r} f_T(r, \hat{X}^{t,x}_r, \hat{M}^{t,u}_r, \hat{\xi}_r)dr].$$

From Lemma 5.4, $\hat{\xi} \in \mathbb{A}^{\xi}$, and hence Lemma 5.3 can be applied. By sending s to T, we then deduce

$$\phi(t,x,u) \leq \mathbb{E}[\int_t^{s} e^{-\delta(r-t)} f_T(r, \hat{X}^{t,x}_r, \hat{M}^{t,u}_r, \hat{\xi}_r)dr] \leq w(x,u).$$

Combining with the conclusion to (i), this shows that $\phi = w$ on $\mathcal{X}_T$, and that the process

$$\{\hat{\xi}_s = \xi^*(s, \hat{X}^{t,x}_s, \hat{M}^{t,u}_s),\ s \in [t,T]\}$$

is an optimal control.

Now, from Lemma 5.5, we get that that if ξ^1 and ξ^2 are both optimal controls, then

$$\int_0^T e^{-\delta t}|\xi_t^1 - \xi_t^2|^2 dt = 0,$$

hence the optimal control is unique, up to t-almost everywhere and almost sure equivalence.

Conclusion for the initial optimization program We can deduce the unique optimal control $(\hat{c}, \hat{v})$ to the equivalent program $\hat{w}$ from the following system:

$$\begin{cases} \kappa_c \hat{c}_s + \kappa_v \hat{v}_s - \beta \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr = 0, \\ \hat{\xi}_s = \frac{2}{\bar{\kappa}} \left(A(s)\hat{X}_s^{t,x} + B(s) \right) - \frac{\beta}{\kappa_v} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr, \end{cases}$$

so that, for all $s \in [t, T]$, $(q, p, u) \in \mathcal{Y}$,

$$\hat{v}_s = \frac{1}{\kappa_v} \left(\beta \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr - B(s) - A(s)(\hat{E}_s^{t,q} - \hat{V}_s^{t,p}) \right), \quad \hat{c}_s = \frac{1}{\kappa_c} \left(B(s) + A(s)(\hat{E}_s^{t,q} - \hat{V}_s^{t,p}) \right),$$

with $(\hat{E}^q, \hat{V}^p, \hat{M}^u)$ solutions of (5.33) when the strategy $(\hat{c}, \hat{v})$ is employed. □

Lemma 5.1. *If $\xi \in \mathbb{A}_T^\xi$, then for all $(t, x, u) \in \mathcal{X}_T$,*

$$\mathbb{E}\left[\int_t^T e^{-\delta s} \left(\int_t^s |X_u^x|^2 du \right) ds \right] < \infty.$$

Moreover, $\forall s \in [t, T], \quad \mathbb{E}[|M_s^{t,u}|] < \infty$.

Proof. (i) By integration by parts,

$$\begin{aligned} \mathbb{E}\left[\int_t^T e^{-\delta s} \left(\int_t^s |X_r^{t,x}|^2 dr \right) ds \right] &= \mathbb{E}\left[\int_t^T \left(\int_s^T e^{-\delta r} dr \right) |X_s^{t,x}|^2 ds \right] \\ &\leq \frac{1}{\delta} \mathbb{E}\left[\int_t^T e^{-\delta s} |X_s^{t,x}|^2 ds \right]. \end{aligned}$$

Now, referring to the explicit expression of $X^{t,x}$ in (5.35), we have, using Jensen inequality,

$$\begin{aligned} |X_s^{t,x}|^2 &\le 3\left(\left(\mathcal{E}_s^t\right)^2|x|^2 + \int_t^s dr \int_t^s (\mathcal{E}_s^r)^2|\xi_r|^2 dr + z^2\left(\int_t^s \mathcal{E}_s^r dW_r\right)^2\right) \\ &\le 3\left(|x|^2 + \frac{1}{a}\int_t^s |\xi_r|^2 dr + z^2\left(\int_t^s \mathcal{E}_s^r dW_r\right)^2\right). \end{aligned} \tag{5.41}$$

Noting that

$$\mathbb{E}\left[\left(\int_t^s \mathcal{E}_s^r dW_r\right)^2\right] = \mathbb{E}\left[\int_t^s (\mathcal{E}_s^r)^2 dr\right] \le s-t, \tag{5.42}$$

we get

$$\mathbb{E}\left[|X_s^{t,x}|^2\right] \le 3\left(|x|^2 + \frac{1}{a}\int_t^s |\xi_r|^2 dr + z^2(s-t)\right). \tag{5.43}$$

Hence, applying Fubini,

$$\mathbb{E}\left[\int_t^T e^{-\delta s}|X_s^{t,x}|^2 ds\right] \le \tilde{C}\mathbb{E}\left[1 + \int_t^T e^{-\delta s}\left(\int_t^s \xi_r^2 dr\right) dt\right]$$

with a constant $\tilde{C} > 0$. By integration by parts, we have

$$\mathbb{E}\left[\int_t^T e^{-\delta s}\left(\int_t^s \xi_r^2 dr\right) dt\right] \le \frac{1}{\delta}\mathbb{E}\left[\int_t^T e^{-\delta s}\xi_s^2 ds\right],$$

which is finite as $\xi \in \mathbb{A}_T^\xi$. This allows to conclude the first part of the proof.

(ii) Let $s \in [t,T]$. Using the explicit expression of M in (5.36), we have

$$\begin{aligned} \mathbb{E}\left[|M_s^{t,u}|\right] &\le e^{-\rho(s-t)}u + \mathbb{E}\left[\int_t^s e^{-\rho(s-r)}(X_{r-}^{t,x})^2 d\widehat{N}_r\right] \\ &\le u + \frac{2\lambda b^2}{b+1}\int_t^s \mathbb{E}\left[(X_{r-}^{t,x})^2\right] dr. \end{aligned}$$

Now, by Fubini, $\int_t^T e^{-\delta s}\left(\int_t^s \mathbb{E}\left[|X_r^{t,x}|^2\right] dr\right) ds = \mathbb{E}\left[\int_t^T e^{-\delta s}\left(\int_t^s |X_r^{t,x}|^2 dr\right) ds\right]$, which is finite for $\xi \in \mathbb{A}_T^\xi$ according to (i). Thus, $\int_t^T e^{-\delta s}\left(\int_t^s \mathbb{E}\left[|X_r^{t,x}|^2\right] dr\right) ds$ is finite. By the property of the

Lebesgue integral, it implies that $\int_t^s \mathbb{E}\left[|X_r^{t,x}|^2\right] dr$ is finite for $s \in [t,T]$ almost everywhere. Since $s \mapsto \int_t^s \mathbb{E}\left[|X_r^{t,x}|^2\right] dr$ is increasing, $\int_t^s \mathbb{E}\left[|X_r^{t,x}|^2\right] dr$ is actually finite for all $s \in [t,T]$, otherwise a contradiction can be easily exhibited. Hence, for all $s \in [t,T]$, $\mathbb{E}\left[|M_s^{t,u}|\right] < \infty$. This concludes the proof. □

Lemma 5.2. *For any admissible control $\xi \in \mathbb{A}_T^{\xi}$, for all $(t,x,u) \in \mathcal{X}_T$, we have*

$$\mathbb{E}\left[\int_t^T e^{-\delta s}|f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)|ds\right] < \infty.$$

Proof. We have, using that h is inferiorly bounded by a strictly positive term that we write $1/\eta$ with the constant $\eta \in \mathbb{R}_+^*$,

$$\mathbb{E}\left[\int_t^T e^{-\delta s}|f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)|ds\right] \leq \mathbb{E}\left[\int_t^T e^{-\delta s}\left(\eta\beta|X_s^{t,x}| + \alpha M_s^{t,u} + \frac{\bar{\kappa}}{4}\left(\xi_s + \frac{\beta}{\kappa_v}\int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)}dr\right)^2\right) ds\right]$$

Now, from the explicit expression of $X^{t,x}$ in (5.35), it holds:

$$\begin{aligned} |X_s^{t,x}| &\leq \mathcal{E}_s^t|x| + \mathcal{E}_s^t\left|\int_t^s (\mathcal{E}_r^t)^{-1}\left\{\xi_r dr + z dW_r\right\}\right| \\ &\leq |x| + \int_t^s |\xi_r|dr + z\left(1 + \left(\int_t^s \mathcal{E}_s^r dW_r\right)^2\right). \end{aligned} \tag{5.44}$$

By (5.42), we get

$$\mathbb{E}\left[|X_s^{t,x}|\right] \leq \mathbb{E}\left[|x| + \int_t^s |\xi_r|dr + z(1+s-t)\right]. \tag{5.45}$$

Moreover, by integration by parts,

$$\mathbb{E}\left[\int_t^T e^{-\delta s}\left(\int_t^s |\xi_r|dr\right)ds\right] \leq \mathbb{E}\left[\frac{1}{\delta}\int_t^T |\xi_s|ds\right].$$

As $\xi \in \mathbb{A}_T^{\xi}$, the expectation in the right-hand side is finite. Hence,

$$\mathbb{E}\left[\int_t^T e^{-\delta s}|X_s^{t,x}|ds\right] < \infty.$$

As for $\mathbb{E}\left[\int_t^T e^{-\delta s}|M_s^{t,u}|ds\right]$, we have, by the explicit expression of $M^{t,u}$ in equation (5.36):

$$\begin{aligned}\mathbb{E}\left[\int_t^T e^{-\delta s}|M_s^{t,u}|ds\right] &\leq \mathbb{E}\left[\int_t^T e^{-\delta s}\left(e^{-\rho(s-t)}u + \int_0^s e^{-\rho(s-r)}(X_{r-}^{t,x})^2 d\widehat{N}_r\right)ds\right] \\ &\leq \int_t^T e^{-\delta s}\left(u + \frac{2\lambda b^2}{b+1}\int_t^s \mathbb{E}[(X_{r-}^{t,x})^2]dr\right)ds,\end{aligned}$$

which is finite for $\xi \in \mathbb{A}^{\xi}$ according to Lemma 5.1.

Finally, $\mathbb{E}\left[\int_t^T e^{-\delta s}\left(\xi_s + \frac{\beta}{\kappa_v}\int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)}dr\right)^2 ds\right] \leq \mathbb{E}\left[\int_t^T e^{-\delta s}\left(\xi_s^2 + \left(\frac{\beta}{\kappa_v}\int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)}dr\right)^2\right)ds\right]$, which is finite as $\xi \in \mathbb{A}_T^{\xi}$. □

Lemma 5.3. *For every $\xi \in \mathbb{A}^{\xi}$ and every $(t,x,u) \in \mathcal{X}_T$,*

$$\mathbb{E}[\sup_{t\leq r\leq s}|\phi(X_r^{t,x}, M_r^{t,u})|] < \infty.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq r\leq s}|\phi(X_r^{t,x}, M_r^{t,u})|\right] \leq \frac{1}{2}A\mathbb{E}\left[\sup_{t\leq r\leq s}(X_r^{t,x})^2\right] + B\mathbb{E}\left[\sup_{t\leq r\leq s}|X_r^{t,x}|\right] \\ + |C|\left(u + \mathbb{E}\left[\sup_{t\leq r\leq s}\int_t^r e^{-\rho(r-y)}(X_{y-}^{t,x})^2 d\widehat{N}_y\right]\right).\end{aligned}$$

Referring to (5.41) and using Burkholder-Davis-Gundy inequality, there exists a positive constant $\tilde{C}$ so that for every $t \geq 0$,

$$\mathbb{E}\left[\sup_{t\leq r\leq s}(X_r^{t,x})^2\right] \leq \tilde{C}\mathbb{E}\left[|x|^2 + \int_t^s |\xi_r|^2 dr + z^2(s-t)\right].$$

This upper boundary is finite as $\xi \in \mathbb{A}_T^{\xi}$. Thus, $\mathbb{E}\left[\sup_{t\leq r\leq s}(X_r^{t,x})^2\right]$ is finite.

Moreover, recalling (5.44), and applying again Burkholder-Davis-Gundy inequality, we similarly get that $\mathbb{E}\left[\sup_{t\le r\le s}|X_r^{t,x}|\right]<\infty$ for $\xi\in\mathbb{A}_T^{\xi}$.

Finally, as $r\mapsto\int_t^r e^{\rho y}(X_{y-}^{t,x})^2 d\widehat{N}_y$ is increasing for each trajectory, we have

$$\mathbb{E}\left[\sup_{t\le r\le s}\int_t^r e^{-\rho(r-y)}(X_{y-}^{t,x})^2 d\widehat{N}_y\right]\le\mathbb{E}\left[\sup_{t\le r\le s}\int_t^r e^{\rho y}(X_{y-}^{t,x})^2 d\widehat{N}_y\right]\le\frac{2\lambda b^2}{b+1}\mathbb{E}\left[\int_t^s e^{\rho y}(X_{y-}^{t,x})^2 dy\right],$$

which is finite since M is integrable for admissible strategies by Lemma 5.1.

Therefore, we can conclude by a finite sum of finite terms that $\mathbb{E}\left[\sup_{0\le s\le t}|\phi(X_s^x,M_s^u)|\right]<\infty$. □

Lemma 5.4. *The optimal control is admissible, i.e.* $\hat{\xi}\in\mathbb{A}_T^{\xi}$.

Proof. As $\forall s\in[t,T]$, $\hat{\xi}_s=\frac{2}{\bar{\kappa}}\Big(A(s)\hat{X}_s^{t,x}+B(s)\Big)-\frac{\beta}{\kappa_v}\int_s^T\frac{e^{-\delta(r-s)}}{h(m_r)}dr$, the explicit solutions for $\hat{X}^{t,x},\hat{M}^{t,u}$ are as follows, for $s\in[t,T]$:

$$\hat{X}_s^{t,x}=\hat{\mathcal{E}}_s^t x+\hat{\mathcal{E}}_s^t\int_t^s(\hat{\mathcal{E}}_r^t)^{-1}\left\{\nu_r dr+z dW_r\right\},\tag{5.46}$$

$$\hat{M}_s^{t,u}=e^{-\rho(s-t)}u+\int_t^s e^{-\rho(s-r)}(\hat{X}_{r-}^{t,x})^2 d\widehat{N}_r,\tag{5.47}$$

with $\nu_s:=\frac{2}{\bar{\kappa}}B(s)-\frac{\beta}{\kappa_v}\int_s^T\frac{e^{-\delta(r-s)}}{h(m_r)}dr$, and $\hat{\mathcal{E}}_s^t=e^{-\int_t^s(a-\frac{2}{\bar{\kappa}}A(r))dr}\prod_{t\le r\le s}(1-\Delta\widetilde{N}_r)$, and writing $0^0=1$. Note that $\forall s\in[t,T]$, $a-\frac{2}{\bar{\kappa}}A(s)\ge 0$, as A is negative.

Now, using the explicit expression of $\hat{X}^{t,x}$ above, equations (5.41) and (5.42), and Fubini, we have

$$\begin{aligned}\mathbb{E}[\int_t^T|\hat{\xi}_s|^2 ds]&\le\tilde{C}\left(1+\mathbb{E}\left[\int_t^T|\hat{X}_s^{t,x}|^2 ds\right]\right)\\&\le\tilde{C}\left(1+\mathbb{E}\left[\int_t^T\left(|x|^2+\int_t^s\nu_r^2 dr+z^2(s-t)\right)ds\right]\right)<\infty,\end{aligned}$$

with a positive constant $\tilde{C}$.

Moreover, $\hat{\xi}$ is $\mathbb{F}$-progressively measurable as $\forall s\in[t,T]$, $\hat{\xi}_s=g((\hat{X}_r^{t,x})_{t\le r\le s})$, with g a continuous function. Hence, $\hat{\xi}$ is an admissible strategy, i.e. is $\in\mathbb{A}_T^{\xi}$. □

Lemma 5.5. *For every $\xi \in \mathbb{A}_T^\xi$, $\forall (t,x,u) \in \mathcal{X}_T$, the functional*

$$\mathcal{J} : (t,x,u,\xi) \mapsto \mathbb{E}\left[\int_t^T e^{-\delta s}\left(-f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)\right) ds\right]$$

is strictly convex in ξ and for $\theta \in [0,1]$, and for $\xi^1, \xi^2 \in \mathbb{A}_T^\xi$,

$$\theta \mathcal{J}(t,x,u,\xi_1) + (1-\theta)\mathcal{J}(t,x,u,\xi_2) - \mathcal{J}(t,x,u,\theta\xi_1 + (1-\theta)\xi_2) \geq \frac{\bar{\kappa}}{4}\int_t^T e^{-\delta s}|\xi_s^1 - \xi_s^2|^2 ds.$$

Proof. Let us show that $\forall s \in [t,T]$, $\mathbb{E}[-f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)]$ is convex in ξ. By linearity of integrals and applying Fubini thanks to Lemma 5.2, it will be so for $\mathcal{J}$. We have

$$\mathbb{E}\left[-f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)\right] = -\frac{\beta}{h(m_s)}\mathbb{E}\left[X_s^{t,x}\right] + \alpha \mathbb{E}\left[M_s^{t,u}\right] + \mathbb{E}\left[\frac{\bar{\kappa}}{4}\left(\xi_s + \frac{\beta}{\kappa_v}\int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr\right)^2\right].$$

Now, $X_s^{t,x}$ is linear in ξ according to its explicit expression (5.35). The last term is obviously strictly convex in ξ_s. As for $M_s^{t,u}$, using its explicit expression (5.36) and the properties of admissible strategies ($\in \mathbb{A}_T^\xi$),

$$\mathbb{E}[M_s^{t,u}] = e^{-\rho(s-t)}u + \frac{2\lambda b^2}{1+b}\mathbb{E}\left[\int_t^s e^{-\rho(s-r)}(X_r^{t,x})^2 dr\right].$$

Now, $(X_r^{t,x})^2$ is strictly convex in ξ by Jensen inequality. Therefore, by addition of linear and strictly convex terms in ξ, $\mathbb{E}[-f(s, X_s^{t,x}, M_s^{t,u}, \xi_s)]$ is strictly convex in ξ, and so is $\mathcal{J}$. More precisely, by focusing only on the third part, it is easy to show that for $\theta \in [0,1]$, and for $\xi^1, \xi^2 \in \mathbb{A}_T^\xi$,

$$\theta \mathcal{J}(t,x,u,\xi_1) + (1-\theta)\mathcal{J}(t,x,u,\xi_2) - \mathcal{J}(t,x,u,\theta\xi_1 + (1-\theta)\xi_2) \geq \frac{\bar{\kappa}}{4}\int_t^T e^{-\delta s}|\xi_s^1 - \xi_s^2|^2 ds.$$

□

Existence and uniqueness of the mean field equilibrium In the next Proposition, we show that there exists a unique mean field equilibrium to the Greenwashing MFG, when the function h is increasing and admits a positive lower bound.

Proposition 17 (Existence and uniqueness of the MFE)**.** *Assume that the function h is increasing, and that there exists $\eta > 0$ so that for all $x \in \mathbb{R}$, $h(x) \geq \frac{1}{\eta}$. Then, there exists a unique mean field equilibrium to the Greenwashing mean field game.*

Proof of Proposition 17. This proof is conducted in three steps. In (i), we specify a functional, $\Psi : \mathcal{C}^1([0,T],\mathbb{R}) \mapsto \mathcal{C}^1([0,T],\mathbb{R})$, of which the fixed point(s) characterize the MFE of the Greenwashing MFG. In (ii), we show that this functional admits at least one fixed point, which means that this MFG admits at least one MFE. In (iii), we show that, if the greenwashing MFG admits a MFE, it must be unique. Together, (ii) and (iii) prove the result stated in this Proposition.

(i) Let us define the following map:

$$\Psi : \mathcal{C}^1([0,T],\mathbb{R}) \ni m \mapsto (\Psi_t(m))_{0\leq t\leq T} \in \mathcal{C}^1([0,T],\mathbb{R}),$$

with, for all $t \in [0,T]$,

$$\Psi_t(m) := g_t(m) + p + \frac{1}{\kappa_v}\int_0^t \left[\beta f_s(m) - B_s(m) - A_s g_s(m)\right] ds, \tag{5.48}$$

and for the functions $f_t, B_t, g_t : \mathcal{C}^1([0,T],\mathbb{R}) \to \mathcal{C}^1([0,T],\mathbb{R})$ defined as follows:

$$f_t(m) := \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds, \qquad B_t(m) := \beta\int_t^T e^{-\int_t^s(\zeta_u+\delta)du}\left(\frac{1}{h(m_s)} - \frac{A(s)}{\kappa_v} f_s(m)\right) ds,$$

$$g_t(m) = e^{-\int_0^t \zeta_s ds} x + \int_0^t e^{-\int_s^t \zeta_r dr}\left(\frac{2}{\bar{\kappa}} B_s(m) - \frac{\beta}{\kappa_v} f_s(m)\right) ds,$$

writing $\zeta_u := -\frac{2}{\bar{\kappa}}A(u) + a + \lambda b$

Then, let us show that the set of fixed points of Ψ characterize the set of MFE of the Greenwashing mean field game. Assume that there exists $m^* \in \mathcal{C}^1([0,T],\mathbb{R})$ so that $\Psi(m^*) = m^*$. According to Proposition 16, the optimal strategy in response to m^*, written (v^*, c^*), verifies:

$$c_t^* = \frac{1}{\kappa_c}\left(B(t) + A(t)(E_t^* - V_t^*)\right), \qquad v_t^* = \frac{1}{\kappa_v}\left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s^*)} ds - B(t) - A(t)(E_t^* - V_t^*)\right), \tag{5.49}$$

where E^*, V^* are solution to the dynamics (5.29) when the optimal strategy (v^*, c^*) is employed. More generally, let us write any state variable with an index * whenever it is driven by the strategy (v^*, c^*). Let us show that (v^*, c^*, m^*) is a mean field equilibrium. By Proposition 16, the condition (i) of the definition of a MFE is verified. Writing $X^* := E^* - V^*$ in a similar fashion as in the proof of Proposition 16, we get that for any $t \in [0, T]$, $\mathbb{E}[E_t^*] = \mathbb{E}[X_t^*] + \mathbb{E}[V_t^*]$. According to the proof of Proposition 16, equation (5.35), the explicit solution of X^* verifies the following, for $t \in [0, T]$:

$$X_t^* = \hat{\mathcal{E}}_t x + \hat{\mathcal{E}}_t \int_0^t \hat{\mathcal{E}}_s^{-1} \left\{ \left(\frac{2}{\bar{\kappa}} B(s) - \frac{\beta}{\kappa_v} \int_s^T \frac{e^{-\delta(u-s)}}{h(m_u^*)} du \right) ds + z dW_s \right\},$$

with $\hat{\mathcal{E}}_t = e^{\int_0^t (\frac{2}{\bar{\kappa}} A(s) - a) ds} (1-b)^{N_t}$, writing $0^0 = 1$. Hence,

$$\mathbb{E}[X_s^*] = e^{-\int_0^s \zeta_u du} x + \int_0^s e^{-\int_u^s \zeta_r dr} \left(\frac{2}{\bar{\kappa}} B(u) - \frac{\beta}{\kappa_v} \int_u^T \frac{e^{-\delta(r-u)}}{h(m_r^*)} dr \right) du.$$

Moreover, the explicit expression of V^* is the following:

$$V_t^* = p + \int_0^t r_t^* dt = p + \frac{1}{\kappa_v} \int_0^t \left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s^*)} ds - B(t) - A(t) X_t^* \right) dt.$$

Hence,

$$\mathbb{E}[V_t^*] = p + \frac{1}{\kappa_v} \int_0^t \left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s^*)} ds - B(t) - A(t) \mathbb{E}[X_t^*] \right) dt.$$

As a result, we have, by assumption,

$$\mathbb{E}[E_t^*] = g_t(m^*) + p + \frac{1}{\kappa_v} \int_0^t \left[\beta f_s(m^*) - B_s(m^*) - A_s g_s(m^*) \right] ds = \Psi_t(m^*) = m_t^*.$$

Hence, condition (ii) of Definition 6 is verified as well. This means that (v^*, c^*, m^*) is a mean field equilibrium.

(ii) To show that Ψ admits at least one fixed point, we apply Shauder fixed point theorem, restated at the end of Internet Appendix Section 5 for the sake of completeness. Let $K := \mathcal{C}^1([0,T],\mathbb{R})$, normed by $\|.\| : m \in K \mapsto \int_0^T |m_t|dt$. K is a nonempty convex closed subset of a Hausdorff topological vector space, from the properties of real valued continuous functions defined on a compact set. Moreover, Ψ is continuous as it is a linear combination of continuous functions. To show that $\Psi(K)$ is included in a compact subset of K, we use Arzelà-Ascoli theorem, restated at the end of Internet Appendix Section 5 for the sake of completeness. To be able to apply it to our setting, let us show that the set $\Psi(K)$ is (a) uniformly bounded, (b) uniformly equicontinuous.

(a) Let $m \in K$, $t \in [0,T]$. Then, for all $t \in [0,T]$,

$$|\Psi_t(m)| \leq |g_t(m)| + |p| + \frac{1}{\kappa_v}\int_0^t \big(\beta|f_s(m)| + |B_s(m)| + |A_s||g_s(m)|\big)ds.$$

Now, as $\frac{1}{h(x)} \leq \eta$, $\forall x \in \mathbb{R}$, we have $|f_t(m)| = \int_t^T \frac{1}{h(m_u)}du \leq T\eta$.

Using this inequality and similar arguments, we get

$$\begin{aligned}
|B_t(m)| &\leq \beta \int_t^T e^{-\int_t^s \zeta_u du}\left(\frac{1}{h(m_s)} + \frac{|A(s)|}{\kappa_v}|f_s(m)|\right)ds \leq \beta T\eta\left(1 + \frac{1}{\kappa_v}\int_0^T |A(s)|ds\right),\\
|g_t(m)| &\leq e^{-\int_0^t \zeta_u du}|x| + \int_0^t e^{-\int_u^t \zeta_r dr}\left(\frac{2}{\bar{\kappa}}|B_u(m)| + \frac{\beta}{\kappa_v}|f_u(m)|\right)du\\
&\leq |x| + \frac{2}{\bar{\kappa}}\beta T^2\eta\left(1 + \frac{1}{\kappa_v}\int_0^T |A(s)|ds\right) + \frac{\beta}{\kappa_v}T^2\eta.
\end{aligned}$$

Hence,

$$\int_0^t |A_s||g_s(m)|ds \leq \beta T^2\eta \int_0^T |A_s|ds\left(|x| + \frac{2}{\bar{\kappa}}\left(1 + \frac{1}{\kappa_v}\int_0^T |A(s)|ds\right) + \frac{1}{\kappa_v}\right).$$

Summing all these upper boundaries which do not depend on t nor on m, we get an upper boundary for $|\Psi_t(m)|$ which does not depend on t nor on m. Therefore, $\Psi(K)$ is uniformly bounded.

(b) Let us show that $\Psi(K)$ is equicontinuous, i.e. that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall m \in K, \forall (t_1,t_2) \in [0,T]^2, (|t_1 - t_2| \leq \delta \Rightarrow |\Psi_{t_1}(m) - \Psi_{t_2}(m)| \leq \epsilon).$$

Let $m \in K$, $(t_1, t_2) \in [0,T]^2$. We have

$$|\Psi_{t_1}(m) - \Psi_{t_2}(m)| \leq |g_{t_1}(m) - g_{t_2}(m)| + \frac{1}{\kappa_v} \int_{t_2}^{t_1} [\beta |f_s(m)| + |B_s(m)| + |A_s||g_s(m)|]\, ds,$$

with

$$|g_{t_1}(m) - g_{t_2}(m)| \leq (1 - e^{-\int_0^T \zeta_u du})|t_1 - t_2||x| + \int_{t_2}^{t_1} e^{-\int_u^s \zeta_r dr} \left(\frac{2}{\bar{\kappa}} B_u(m) - \frac{\beta}{\kappa_v} f_u(m) \right) du.$$

Hence, using the boundaries established in (a), if we define the constant C as follows,

$$C := \max \Bigg((1 - e^{-\int_0^T \zeta_u du})|x|,\ \frac{2}{\bar{\kappa}} \beta T \eta \left(1 + \frac{1}{\kappa_v} \int_0^T |A(s)| ds \right) + \frac{\beta}{\kappa_v} T \eta,\ \beta T \eta \left(1 + \frac{1}{\kappa_v} \int_0^T |A(s)| ds \right), \sup_{0 \leq s \leq T} |A_s| \left(|x| + \frac{2}{\bar{\kappa}} \beta T^2 \eta \left(1 + \frac{1}{\kappa_v} \int_0^T |A(s)| ds \right) + \frac{\beta}{\kappa_v} T^2 \eta \right) \Bigg),$$

we have

$$|\Psi_{t_1}(m) - \Psi_{t_2}(m)| \leq C|t_1 - t_2|.$$

As C does not depend on m nor on t_1, t_2, $\Psi(K)$ is uniformly equicontinuous.

Hence, by Arzela-Ascoli theorem, we can conclude that $\Psi(K)$ is a compact subset of K. Thus, Shauder fixed point theorem can be applied, and proves that the set of fixed points of the mapping Ψ is non-empty. This means that the Greenwashing MFG admits at least one mean field equilibrium according to (i).

(iii) To show that this MFE is unique, we only need the objective functional, thanks to Lasry-Lions monotonicity condition. Suppose (v^1, c^1, m^1) and (v^2, c^2, m^2) are two mean field equilibria, and suppose they are distinct. Solutions of equation (5.29) are noted (E^1, V^1, M^1) and (E^2, V^2, M^2) when the strategies (v^1, c^1) and (v^2, c^2) are employed respectively. Note that the two associated optimal controls, (v^1, c^1) and (v^2, c^2), must be distinct, as otherwise we would have $m_t^1 = m_t^2$ for each t, according to condition (ii) of Definition 6. Now, because (v^1, c^1) is optimal when m^1

describes the population flow, it certainly outperforms (v^2, c^2), and we have

$$\mathbb{E}\left[\int_0^T \left(\beta \frac{E_t^1}{h(m_t^1)} - \alpha M_t^1 - \frac{\kappa_v}{2}(v_t^1)^2 - \frac{\kappa_c}{2}(c_t^1)^2\right) dt\right] > \mathbb{E}\left[\int_0^T \left(\beta \frac{E_t^2}{h(m_t^1)} - \alpha M_t^2 - \frac{\kappa_v}{2}(v_t^2)^2 - \frac{\kappa_c}{2}(c_t^2)^2\right) dt\right].$$

Similarly,

$$\mathbb{E}\left[\int_0^T \left(\beta \frac{E_t^2}{h(m_t^2)} - \alpha M_t^2 - \frac{\kappa_v}{2}(v_t^2)^2 - \frac{\kappa_c}{2}(c_t^2)^2\right) dt\right] > \mathbb{E}\left[\int_0^T \left(\beta \frac{E_t^1}{h(m_t^2)} - \alpha M_t^1 - \frac{\kappa_v}{2}(v_t^1)^2 - \frac{\kappa_c}{2}(c_t^1)^2\right) dt\right].$$

Adding these two inequalities, and using that $\mathbb{E}[E_t^1] = m_t^1$, $\mathbb{E}[E_t^2] = m_t^2$, we get

$$\beta \int_0^T \left(\frac{m_t^1}{h(m_t^1)} + \frac{m_t^2}{h(m_t^2)} - \frac{m_t^2}{h(m_t^1)} - \frac{m_t^1}{h(m_t^2)}\right) dt > 0.$$

Now, the term inside the integral is equal to

$$\frac{(m_t^1 - m_t^2)(h(m_t^2) - h(m_t^1))}{h(m_t^1)h(m_t^2)},$$

which is non-positive as h is increasing and positive, and negative at least for some Lebesgue-non-negligible set of times t as h is monotone and m^1, m^2 are distinct from one another. Hence, a contradiction is exhibited. This proves uniqueness of the mean field equilibrium. □

Numerical simulation of the mean field equilibrium Finding an analytical expression to the mean field equilibrium seems inaccessible, as the representative company's program is non linear-quadratic. However, thanks to Proposition 16, we are able to express an explicit map, Ψ, from which the unique fixed point is equal to the average environmental rating, m^*, in the mean field equilibrium, (v^*, c^*, m^*). From m^*, the optimal strategy (v^*, c^*) at the MFE can be recovered thanks to Proposition 16. The fixed point of Ψ can be approximated numerically. For this numerical approximation, we use the Fictitious Play algorithm.

Let the best response function, $\hat{\beta} : \mathcal{C}^1([0,T], \mathbb{R}) \to \mathbb{A}_T$, map the optimal strategy $(\hat{v}, \hat{c})$ to a given average environmental rating, m, as given in Proposition 16. Moreover, note

$(E^{(v,c)}, M^{(v,c)}, V^{(v,c)})$ the solution to equation (5.29) when the strategy $(v, c) \in \mathbb{A}_T$ is employed. Then, the map $\Psi : \mathcal{C}^1([0,T],\mathbb{R}) \to \mathcal{C}^1([0,T],\mathbb{R})$ is as follows: $\Psi(m) = (\mathbb{E}[E_t^{\hat{\beta}(m)}])_{0\leq t\leq T}$. Its explicit expression is given in the proof of Proposition 17, equation (5.48).

To approximate the fixed point of Ψ, the Fictitious Play algorithm respects the following iteration rule, for $k \in \mathbb{N}^*$:

$$m_k = \frac{1}{k}\Psi(m_k) + \frac{k-1}{k} m_{k-1}.$$

Perrin, Pérolat, Laurière, Geist, Elie, and Pietquin (2020); Dumitrescu, Leutscher, and Tankov (2023) prove the convergence of this algorithm in similar frameworks. In our framework, we can use the notion of "exploitability" to control for the convergence of our algorithm.

Definition 7 (Exploitability)**.** The exploitability ε_k of the Fictitious Play algorithm at iteration $k \in \mathbb{N}^*$ is equal to

$$\varepsilon_k = J(\hat{\beta}(m_{k-1}), m_k) - J(\hat{\beta}(m_k), m_k),$$

with J the objective functional of the Greenwashing MFG to be minimized (5.31).

The exploitability measures potential improvement for the representative agent from the current iteration. Its interest is related to the notion of an ε-Mean Field Equilibrium, which formalizes the notion of approximate MFE.

Definition 8 (ε-Mean Field Equilibrium)**.** An ε-Mean Field Equilibrium, for an $\varepsilon > 0$, is a triplet $(\hat{v}^\varepsilon, \hat{c}^\varepsilon, \hat{m}^\varepsilon) \in \mathbb{A}_T \times \mathcal{C}^1([0,T],\mathbb{R})$ so that for all $(v, c) \in \mathbb{A}_T$,

$$J(v, c, \hat{m}^\varepsilon) \leq J(\hat{v}^\varepsilon, \hat{c}^\varepsilon, \hat{m}^\varepsilon) + \varepsilon.$$

Note that, by definition, a 0-MFE is a MFE. The exploitability allows to characterize approximate MFE, as shown in the next Proposition.

Proposition 18. *Let ε_k be the exploitability at iteration k of the Fictitious play algorithm. Then, $(\hat{\beta}(m_{k-1}), m_k)$ is an ε_k-mean field equilibrium.*

Proof. We have $\varepsilon_k = J(\hat{\beta}(m_{k-1}), m_k) - J(\hat{\beta}(m_k), m_k)$. Hence, for all $(v, c) \in \mathbb{A}_T$, by definition of the best reponse function $\hat{\beta}$, $\varepsilon_k \geq J(\hat{\beta}(m_{k-1}), m_k) - J(v, c, m_k)$. This means that $(\hat{\beta}(m_{k-1}), m_k)$ is an ε_k-MFE. □

Simulations The algorithm is implemented on the baseline calibration, except for one parameter. Indeed, to allow comparability with the case without interaction, we change β to 50, so that companies have the same incentive to increase their environmental score at the initial date, whether or not their scores are normalized. Time horizon is set to 100, as it is enough to reach some stationary pattern between the initial and terminal conditions. The function h is set as follows: for $x \in \mathbb{R}$, $h(x) = \max(1, x)$. Hence, for $x \geq 1$, h is equal to the identity function. The initial values of the environmental score and the environmental value are set high enough so that the probability that the environmental score fall below 1 is negligible: they are set to 50 each, with a measurement error volatility $z = 0.2$. The initial value of the misrating proxy is set at 5. For the simulations, time is discretized with a time step equal to 10^{-3}. The Fictitious Play algorithm is initialized with a constant vector, m_{init}, equal to 50.

In our Fictitious Play algorithm, for which we run 500 iterations, the exploitability converges to zero very quickly (Figure 5.1). Moreover, graphically, after a few iterations, the curves representing m_k, m_{k+j}, $j \geq 0$, merge perfectly. This suggests that we are approaching very efficiently, and very precisely the MFE.

To interpret the results, we compare the main quantities at the MFE with the ones in the "benchmark" case where there is no interaction between companies. This benchmark case corresponds to the resolution of the Greenwashing program (5.31) for the representative company when the function h is constant equal to 1 and the pro-environmental sensitivity of the investor, β, is equal to 1. This represents the optimum as in Section 3 but with finite horizon, for these results to be comparable with the MFE.

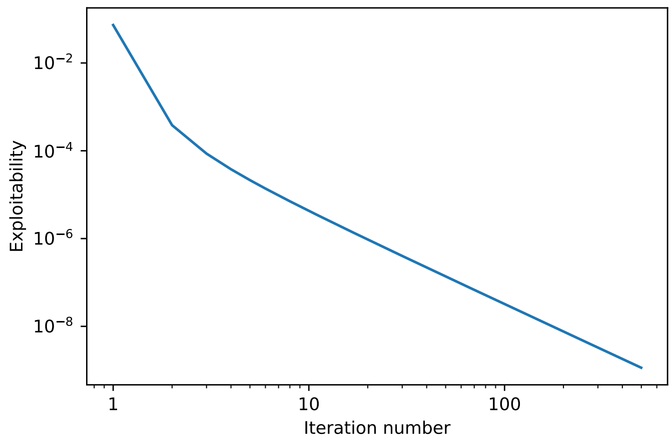

Figure 5.1: Convergence of the exploitability.

Standard theorems

Theorem 19 (Shauder fixed point theorem)**.** *If K is a nonempty convex closed subset of a Hausdorff topological vector space V and f is a continuous mapping of K into itself such that $f(K)$ is contained in a compact subset of K, then the set of fixed points of f is non-empty.*

Theorem 20 (Arzela-Ascoli)**.** *Consider a sequence of real-valued continuous functions $(f_n)_{n\in\mathbb{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges uniformly.*

6 Variables and Tables

Table 6.2: **First-step estimation.** This Table gives the results of the first-step estimation, which is a 2SLS Within panel regression with robust standard errors of the environmental reputation index at the end of the month t, Rep_t, on the environmental controversy index at the end of the month t that is instrumented by the environmental controversy index at the end of the month $t-1$, Con_t^*. The standard deviation is shown in brackets below the estimate.

	Dependent variable: Rep_t
Con_t^*	0.040***
	(0.013)
Firm FE	Yes
Observations	152,821
R^2	0.002
Adjusted R^2	−0.023
F Statistic	240.292***
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 6.3: **Relevance of the instrument used in the second-step estimation (top brownest companies and entire universe).** This table shows the results of the Within regression with robust standard errors of the change in environmental score, ΔE_t^i, on the lagged environmental score, E_{t-1}^i. Both variables are used in the step-2 regression: the former is the independent variable and the latter is the instrument. The estimations are performed for different samples: the top 10%, 20%,..., 90% brownest companies, and the entire universe. The standard deviation is shown in brackets.

	Dependent variable: ΔE_t^i				
	Top brownest companies:				
	10%	20%	30%	40%	50%
E_{t-2}^i	-0.259^{***}	-0.162^{***}	-0.108^{***}	-0.084^{***}	-0.071^{***}
	(0.028)	(0.015)	(0.009)	(0.006)	(0.004)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	19,942	32,667	46,884	60,320	72,470
R^2	0.218	0.123	0.074	0.054	0.044
Adjusted R^2	0.167	0.074	0.028	0.012	0.004
F Statistic	5,224.462***	4,325.124***	3,572.930***	3,310.524***	3,215.831***

	Dependent variable: ΔE_t^i				
	Top brownest companies:				
	60%	70%	80%	90%	Whole sample
E_{t-2}^i	-0.059^{***}	-0.048^{***}	-0.039^{***}	-0.030^{***}	-0.023^{***}
	(0.003)	(0.002)	(0.002)	(0.001)	(0.001)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	88,223	102,884	116,290	130,457	152,821
R^2	0.034	0.026	0.021	0.015	0.011
Adjusted R^2	−0.002	−0.007	−0.010	−0.013	−0.014
F Statistic	2,981.010***	2,644.557***	2,366.927***	1,951.302***	1,673.251***

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table 6.4: **Relevance of the instrument used in the second-step estimation (top greenest companies and entire universe).** This table shows the results of the Within regression with robust standard errors of the change in environmental score, ΔE_t^i, on the lagged environmental score, E_{t-1}^i. Both variables are used in the step-2 regression: the former is the independent variable and the latter is the instrument. The estimations are performed for different samples: the top 10%, 20%,..., 90% greenest companies, and the entire universe. The standard deviation is shown in brackets.

	Dependent variable: ΔE_t^i				
	Top greenest companies:				
	10%	20%	30%	40%	50%
E_{t-2}^i	−0.053***	−0.046***	−0.042***	−0.039***	−0.034***
	(0.005)	(0.003)	(0.002)	(0.002)	(0.002)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	22,364	36,531	49,937	64,598	80,351
R^2	0.041	0.034	0.030	0.027	0.022
Adjusted R^2	0.006	0.002	−0.0001	−0.003	−0.006
F Statistic	917.932***	1,234.863***	1,507.587***	1,723.228***	1,764.086***
	Dependent variable: ΔE_t^i				
	Top greenest companies:				
	60%	70%	80%	90%	Whole sample
E_{t-2}^i	−0.031***	−0.027***	−0.025***	−0.023***	−0.023***
	(0.002)	(0.001)	(0.001)	(0.001)	(0.001)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	92,501	105,937	120,154	132,879	152,821
R^2	0.019	0.016	0.013	0.012	0.011
Adjusted R^2	−0.010	−0.012	−0.014	−0.015	−0.014
F Statistic	1,708.114***	1,643.758***	1,541.759***	1,551.436***	1,673.251***
Note:					*p<0.1; **p<0.05; ***p<0.01

Table 6.5: **Main estimation (top brownest companies and entire universe).** This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$. The estimations are performed for different samples: the top 10%, 20%,..., 90% brownest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

	Dependent variable: $\Delta\hat{c}_t^i$				
Top brownest companies:	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	-0.071	-0.164^{**}	-0.244^{***}	-0.221^{***}	-0.271^{***}
	(0.051)	(0.065)	(0.073)	(0.067)	(0.060)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	18,760	30,711	44,116	56,785	68,276
R^2	0.005	0.006	0.008	0.010	0.013
Adjusted R^2	-0.061	-0.049	-0.041	-0.035	-0.029
F Statistic	0.985	3.525^{*}	5.460^{**}	3.608^{*}	4.949^{**}
Top brownest companies:	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	-0.237^{***}	-0.176^{***}	-0.188^{***}	-0.158^{***}	-0.119^{***}
	(0.053)	(0.049)	(0.046)	(0.040)	(0.033)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	83,309	97,324	110,206	123,864	145,508
R^2	0.015	0.016	0.017	0.017	0.017
Adjusted R^2	-0.023	-0.019	-0.015	-0.012	-0.008
F Statistic	3.476^{*}	1.756	1.875	1.195	0.661

Note: *p<0.1; **p<0.05; ***p<0.01

Table 6.6: **Main estimation (top greenest companies and entire universe).** This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$. The estimations are performed for different samples: the top 10%, 20%,..., 90% greenest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

	Dependent variable: $\Delta\hat{c}_t^i$				
Top greenest companies:	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	-0.255^{***}	-0.342^{***}	-0.446^{***}	-0.405^{***}	-0.415^{***}
	(0.079)	(0.069)	(0.072)	(0.061)	(0.057)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	21,644	35,302	48,184	62,199	77,232
R^2	0.018	0.019	0.021	0.020	0.020
Adjusted R^2	−0.018	−0.013	−0.010	−0.010	−0.009
F Statistic	4.284^{**}	8.542^{***}	14.584^{***}	11.377^{***}	10.606^{***}
Top greenest companies:	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	-0.404^{***}	-0.380^{***}	-0.294^{***}	-0.237^{***}	-0.119^{***}
	(0.052)	(0.054)	(0.052)	(0.044)	(0.033)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	88,723	101,392	114,797	126,748	145,508
R^2	0.022	0.022	0.022	0.021	0.017
Adjusted R^2	−0.007	−0.006	−0.006	−0.006	−0.008
F Statistic	8.727^{***}	6.709^{***}	3.513^{*}	2.169	0.661
Note:					*p<0.1; **p<0.05; ***p<0.01

Table 6.7: **Main estimation with time fixed effects and controls (top brownest companies and entire universe).** This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, including time fixed effects as well as controls for systematic risk and return. The estimations are performed for different samples: the top 10%, 20%,..., 90% brownest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

	Dependent variable: $\Delta\hat{c}_t^i$				
	Top brownest companies:				
	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	0.168	−0.150	−0.253**	−0.241***	−0.459***
	(0.161)	(0.136)	(0.128)	(0.087)	(0.159)
R_{t-1}^i	0.216	0.135	0.324*	0.180	0.139
	(0.260)	(0.182)	(0.197)	(0.140)	(0.145)
$\beta_{t-1}^{CAPM,i}$	0.012	0.038**	−0.010	0.009	0.021*
	(0.023)	(0.016)	(0.021)	(0.012)	(0.013)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	6,044	9,190	12,473	15,507	18,033
R^2	0.006	0.008	0.013	0.014	0.019
Adjusted R^2	−0.073	−0.056	−0.044	−0.037	−0.028
F Statistic	1.020	1.575	2.704	1.828	4.724
	Dependent variable: $\Delta\hat{c}_t^i$				
	Top brownest companies:				
	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	−0.281**	−0.195*	−0.164*	−0.166**	−0.083*
	(0.130)	(0.105)	(0.091)	(0.072)	(0.050)
R_{t-1}^i	0.188	0.366**	0.374**	0.322**	0.252**
	(0.147)	(0.163)	(0.154)	(0.137)	(0.124)
$\beta_{t-1}^{CAPM,i}$	0.012	0.021	0.009	0.012	0.010
	(0.011)	(0.014)	(0.010)	(0.008)	(0.007)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	21,749	25,249	28,980	33,168	41,252
R^2	0.018	0.017	0.017	0.019	0.016
Adjusted R^2	−0.025	−0.023	−0.019	−0.013	−0.012
F Statistic	2.420	4.896	4.795	4.225	3.014

Note: *p<0.1; **p<0.05; ***p<0.01

Table 6.8: **Main estimation with time fixed effects and controls (top greenest companies and entire universe).** This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta \hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, including time fixed effects as well as controls for systematic risk and return. The estimations are performed for different samples: the top 10%, 20%,..., 90% greenest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

	Dependent variable: $\Delta \hat{c}_t^i$				
	Top greenest companies:				
	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	−0.205	−0.380**	−0.261*	−0.243**	−0.280***
	(0.182)	(0.178)	(0.142)	(0.096)	(0.093)
R_{t-1}^i	−0.335	−0.222	−0.002	0.348	0.480**
	(0.287)	(0.245)	(0.217)	(0.241)	(0.232)
$\beta_{t-1}^{CAPM,i}$	0.005	0.008	−0.013	0.008	−0.009
	(0.015)	(0.014)	(0.027)	(0.013)	(0.014)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	8,084	12,272	16,003	19,503	23,219
R^2	0.016	0.021	0.023	0.022	0.020
Adjusted R^2	−0.023	−0.012	−0.008	−0.009	−0.009
F Statistic	1.504	3.582	1.748	3.120	5.449
	Dependent variable: $\Delta \hat{c}_t^i$				
	Top greenest companies:				
	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	−0.385***	−0.284***	−0.251***	−0.193***	−0.083*
	(0.093)	(0.086)	(0.093)	(0.067)	(0.050)
R_{t-1}^i	0.375*	0.185	0.316*	0.255*	0.252**
	(0.220)	(0.170)	(0.171)	(0.153)	(0.124)
$\beta_{t-1}^{CAPM,i}$	0.005	0.008	−0.011	−0.0002	0.010
	(0.011)	(0.011)	(0.012)	(0.010)	(0.007)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	25,745	28,779	32,062	35,208	41,252
R^2	0.023	0.022	0.023	0.022	0.016
Adjusted R^2	−0.007	−0.007	−0.006	−0.006	−0.012
F Statistic	5.711	2.722	4.029	2.754	3.014

Note: *p<0.1; **p<0.05; ***p<0.01

Table 6.9: **Main estimation with different starting dates.** This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, from different starting dates. This estimation is performed on the whole sample. The standard deviations are shown in brackets below the estimates.

	Dependent variable: $\Delta\hat{c}_t^i$			
	Since 2012	Since 2017	Since 2019	Since 2021
$\Delta E_t^{i,*}$	-0.119^{***}	-0.130^{***}	-0.186^{***}	-0.198^{***}
	(0.033)	(0.036)	(0.044)	(0.051)
Firm FE	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes
Observations	145,508	122,345	91,107	39,866
R^2	0.017	0.019	0.023	0.025
Adjusted R^2	−0.008	−0.012	−0.018	−0.072
F Statistic	0.661	0.890	3.311^{*}	5.038^{**}

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table 6.10: **Main estimation applied to different environmental subscores.** This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, applied to different environmental subscores, which are related to (i) the environmental impacts of the products sold ($E_t^{Imp,i,*}$), (ii) the resources used ($E_t^{Res,i,*}$), and (iii) the emissions, effluents, and waste ($E_t^{Emi,i,*}$). This estimation is performed on the whole sample. The standard deviations are shown in brackets below the estimates.

	Dependent variable: $\Delta\hat{c}_t^i$		
	(1)	(2)	(3)
$\Delta E_t^{Imp,i,*}$	-0.070^{***}		
	(0.025)		
$\Delta E_t^{Res,i,*}$		-0.075^{***}	
		(0.021)	
$\Delta E_t^{Emi,i,*}$			-0.058^{***}
			(0.022)
Firm FE	Yes	Yes	Yes
Time FE	Yes	Yes	Yes
Observations	145,508	145,508	145,508
R^2	0.010	0.007	0.014
Adjusted R^2	−0.016	−0.018	−0.012
F Statistic	0.420	0.656	0.401
Note:	*p<0.1; **p<0.05; ***p<0.01		

Table 6.11: **Variables.** Description of and statistics on the variables used in the empirical analysis.

Variable	Mean	Std. Dev.	Min.	Max.	# values	Description
Rep	83.0	18.7	0	100	155,852	Rep is the monthly environmental reputation score based on forward-looking news data, provided by Covalence, "reflecting companies' sustainability commitments, targets, and ambitions" regarding the environment. "The basic metrics used are quantities of news items gathered on the web that can be coded as having a positive or negative polarity towards named companies. Positive news articles are called "endorsements," while articles with negative polarity are "controversies." A historical erosion factor is applied to the quantities of positive and negative news with recent articles weighting more than older ones. The sentiment, or reputation score, is given by the share of positive news over the total of positive and negative news."
Con	8.7	19.6	0	100	155,707	Con is the monthly environmental controversy score provided by Covalence. It is constructed in a similar way to the environmental reputation score using news-based data focused on environmental controversies.
E	54.9	10.3	17.9	94.0	155,707	E is the monthly environmental score provided by Covalence. It is constructed based on two types of data: quantitative indicators (data disclosed annually by companies) and news-based data (published by the media and other stakeholders).
ΔE	0.10	0.7	-22.0	21.0	152,821	ΔE is the change in environmental score over two consecutive months.
c	0.2	2.9	-67.9	73.0	149,136	c is the monthly environmental communication variable, which is constructed through the first step of the empirical analysis in Section 4.1.
Δc	-0.004	4.0	-79.0	101.3	145,508	Δc is the change in environmental communication over two consecutive months.
R	0.01	0.1	-0.9	8.8	44,825	R is the monthly realized return on the stocks issued by the companies.
β^{CAPM}	1.1	1.1	-16.7	26.1	44,067	β^{CAPM} is the CAPM beta of each stock defined as $\beta_t^{CAPM,i} = Var^{-1}(v_t^m)Cov(v_t^i, v_t^m)$, where v^i and v^m denote firm i's return and the market return, respectively.

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