Market-consistent valuation: a step towards calculation stability

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Abstract

In this paper we address some of the stability issues raised by the European life insurance regulation valuation scheme. Via an in-depth study of the so-called economic valuation framework, shaped through the market-consistency contract we first point out the practical interest of one of the El Karoui, Loisel, Prigent & Vedani (2017) propositions to enforce the stability of the cut-off dates used as inputs to calibrate actuarial models. This led us to delegitimize the argument of the no-arbitrage opportunity as a regulatory criteria to frame the valuation, and as an opposition to the previously presented approach. Then we display tools to improve the convergence of the economic value estimations be it the VIF or the *SCR*, using usual variance reduction methods and a specific work on the simulation seeds. Through various implementations on a specific portfolio and valuation model we decrease the variance of the estimators by over 16 times.

Key words European regulation, life insurance, no-arbitrage opportunity, economic valuation convergence, market-consistency,

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Introduction

Dating back to the the late 1990s and 2000s,the European Commission began creating a path to enforce the insurance European market in a direction ensuring more protection for the consumer thanks to common and harmonised rules to characterise the solvency of a firm (Solvency I then Solvency II Directives).

The regulatory framework considered consumer protection, based on market and commercial practices. This objective led to one of the two sides of the harmonisation. This article deals with the other side, the implications of defining a common framework for the economic/regulatory capital, which is the main ingredient to explicit the notion of solvency.

Solvency II being risk-based, the need for a comparable measure of the different exposures, be it on the asset or liability sides, led naturally to the concept of "fair value" which was developed in a parallel path (see in particular Thérond (2016) and chapter 2 of Laurent et al. (2016)). Indeed, since the seventies, the notion of "Market price" has been used and refined especially in areas where a transparent and fair trading price is known. For this notion to be adapted to more complex situations, especially when financial options are involved, maiden theoretical works (Black & Scholes (1973), Merton (1973)...) developed the risk-neutral valuation principles framework allowing for those calculations. The accountant community also developed notions voluntarily in line through the definition of "fair value measurement" (see International Accounting Standards Board (2011)).

In practice, though estimating the value of an asset, especially if it is listed, is quite straightforward, it is not the case for insurance companies' liability side.Since life insurance contracts are not exchanged, assigning a fair price to them involves a complex model and its cash flows' projection, even in the Standard Formula case.

This complexity and the regulatory necessity to use economic scenarios on the one hand and, on the other hand, assess regulatory valuations through Monte-Carlo estimations, renders the stability of inputs and the convergence character essential for its use as an efficient tool to quantify solvency and ease the work of European control authorities. These are the two main lines of this paper.

In Section 1 we review briefly the different historical and regulatory aspects which led to the Solvency II characteristics highlighting those with material impact on the market-(in)consistency. Through this Section we underline its need for more stability. Finally we focus on one approach developed in El Karoui et al. (2017) to improve the stability of economic scenarios calibration inputs and lower the manipulability of the current practical scheme. But this method is often opposed by the no-arbitrage opportunity condition, a regulatory requirement for these valuation. This led us to recall this condition basics, operational requirements, and finally to show its incompatibility with the life insurance regulation in Section 2. According to the authors, these developments promote the real practical interest of the aforementioned approach.

In Section 3 we consider the convergence issue for the regulatory estimators. We present and implement some practitioners-oriented approaches to improve their various estimations, of Values of In-Force and Solvency Capital Requirements. In Section 4 we finally propose and implement an aggregated approach (still relatively easy to implement forpractitioners) which has led us to improve Value of In-Force estimation by more than 16×.

1 Market-consistent life insurance valuation in Solvency II - stability issues and first proposition

Mixing fair-value and regulatory valuation led the regulator (through the Solvency II directive - European Commission (2009)) to develop an insurance-adjusted valuation scheme based on financial market practices, the so-called *market-consistency*.

1.1 Market-consistency: financial stability and accounting information comparability

With Solvency II, the European Commission desired to reinforce the European insurance market so that solvency appreciation is made comparable across Europe, ensuring both a level playing field for the companies and the same consumer protection for the insureds and/or policyholders. In this regard, one of the biggest challenges was to ensure a comparable prudential balance sheet template could be populated across Europe notwithstanding the different national accounting norms. However, despite this objective to increase the level of harmonisation, the reserving mechanisms which are driving the estimation of technical provisions still largely depend on country-specific "Solvency I"-GAAP norms. Indeed as all taxes and social contributions are generally based on those national accounting norms, the historical/booking value of assets and liabilities then play the same role as in the Solvency I framework. Therefore, comparing the different values for technical provisions can be quite challenging for the insureds or supervisory authority throughout the whole EU market.

In parallel with the development of solvency directives, the notion of "Fair value" emerged through the discussion at the International Accounting standards Board (IASB) as a way to tackle those comparability issues. In a nutshell, fair value can be characterised in three points (see Thérond (2016)):

1. Level 1: the price of the products is directly listed in an active market
2. Level 2: Parameters or prices can be inferred via a model (such as a financial option, and the implied volatility (IV) of the underlying asset) ("Mark to Market")
3. Level 3: The price is inferred from a model where interactions are necessary to estimate the figure as a consequence of the model ("Mark to Model")

Even if liabilities can actually be exchanged between insurance companies, this does not happen quite often and, even worse, there are no listed market for these trades, be it indexes or reporting of the effective transaction value. The only possible choice of the exposed alternative is the level 3: estimating the value of those via a probability-weighted average of future cash-flows, discounted with a Risk-Free Rate as outlined in the article 77 of European Commission (2009). With such a procedure, at least on a theoretical point of view, all technical provisions are made comparable independently of any national insurance sector's characteristics. As a consequence, the solvency appreciation with respect to own funds and solvency of any EU insurance company is done on the same terms.

However, experts are now backing up from fair valuation because of its inherent procyclicality. Despite the merits regarding comparability, this valuation methods tends to minimise the value of liabilities when markets are bullish and over-estimate them in times of crisis. (see Laux & Leuz (2009), Laux & Leuz (2010), Plantin et al. (2008*a*) and Plantin et al. (2008*b*)).

1.2 Insurance market-consistency - from regulations to practice

Since the only available technique to evaluate liabilities is the mark to model one, an insurance company needs to describe its asset-liabilities management strategy for which the following are needed:

1. a unique risk-free rate spot curve;
2. a methodology to produce future cash flows reflecting unequivocally the policyholders' behaviour;
3. economic scenarios deriving from the initial risk-free rate describing the future evolution of the assets;
4. a set of rules defining the interactions between the scenarios' tables and the policies to browse each sample path's scenario to ultimately produce cash flows out of any policy.

The only intrinsic part of this process is the determination of the risk-free rate spot curve. The regulatory framework from the Directive to the ITS define what is the basic risk-free interest rate term structure According to Article 77a of the Solvency II Directive the relevant risk-free interest rate term structure should be based on relevant financial instruments traded in deep, liquid and transparent (DLT) markets. This provision is further specified in recital 21, Article 1(32), (33) and (34), and Articles 43, 44 and 46 of the Delegated Regulation European Commission (2015*b*). The identification of the relevant financial instruments is based on this DLT assessment which is a list, for each currency, of the maturities for which the market of the relevant financial instrument is considered DLT including the identification of the last maturity for which rates can be observed in DLT markets (section 7.B refers to the determination of the Last Liquid Point (LLP)). On top of those elements, the determination of the Ultimate Forward Rate (UFR), recently provided by EIOPA[1] is key. Indeed, EIOPA provides the EU insurance market a basic risk-free rate (RFR) curve which goes up to 150 years, thanks to interpolation from the LLP to the 60th year for Euro, via Smith Wilson extrapolation and interpolation.

To go from the basic RFR to the one used for calculation, one needs to add the effect of the Credit Risk Adjustment on the one hand and, on the other hand all the relevant Long Term Guarantee package items which can distort the produced interest rate curve such as the volatility adjustment (VA) or the Matching Adjustment (MA). Only with all such information taken into account, insurance companies or their information providers will set up their calibration process to produce their scenario tables. The potential large variation only due to calibration does induce great sensitivity to the final value of both BEL and own funds. This illustrates the principle from Thérond (2016) "one fair value principle, several valuation techniques"

[1]The specifications for the determination of the UFR's level have been released in 2017 ("EIOPA-BoS-17/072") and can be retrieved on EIOPA's website: https://eiopa.europa.eu/Publications/Reports/Specification%20of%20the%20methodology%20to%20derive%20the%20UFR.pdf

1.3 A first proposition for deeper calibration data stability

From most operational practices, the main issue (though not only) of the valuation scheme is the fact most insurance companies use different calibration inputs for their model. This leads to both an instability of valuations in space (the space of potential inputs at a fixed date) and in time (linked to the high volatility of these market inputs). A particular aspect of these problems is linked to the end-of-year regulatory valuation date. This date is the accounting closure date for most listed companies and the financial markets undergo numerous trades for the sake of accounting optimisation. In particular, this leads to irrational market changes ("turn-of the year effect", well-known by traders, see *e.g.* Ritter (1988)), and it is very hazardous to pick market data up during this period being asset prices or other IV. In addition, having the life insurance whole market regulation depend on one single date renders easier possible manipulations (even the Libor rate can be more or less manipulated, see in particular Abrantes-Metz et al. (2012)).

For such reasons we have focused on a specific tool proposed by El Karoui et al. (2017) to improve the fairness of prices obtained through insurance regulation valuations, decrease the manipulability of the scheme and add some stability through inertia to the interest rates models calibration data: the use of monthly-averaged IV matrices for calibration.

In subsection 2.2.3 of their paper, El Karoui et al. shape an *adapted market-consistent constraint* for interest rate models calibration (under the RFR yield curve) on different swaption log-normal IV matrices, in particular averaged matrices so as to insert inertia in calibration data (lower time-dependency).

- The IV averaged on the whole month of October 2017. Only the receiver swaptions of maturity 5, tenors 1 to 10, and the receiver swaptions of tenor 5, maturities 1 to 10.
- The IV averaged on the whole two months of October and November 2014. Only the receiver swaptions of maturity 5, tenors 1 to 10, and the receiver swaptions of tenor 5, maturities 1 to 10.
- The IV as of end-of-year 2014. Only the receiver swaptions of maturity 10, tenors 1 to 10, and the receiver swaptions of tenor 10, maturities 1 to 10 (v2).
- And the IV as of end-of-year 2014. Only the receiver swaptions of maturity 5, tenors 1 to 10, and the receiver swaptions of tenor 5, maturities 1 to 10 (v2).

The four calibrated then simulated tables embed 1,000 economic scenarios without any use of variance reduction techniques, obtained from the same simulation seed. The interest rates model considered is a Libor Market Model, often used by French practitioners at that time. They are used to valuate three standard saving products[2] economic own funds[3]. Results are summarized in table 1.

This implementation has shown the high volatility of European economic valuation scheme. One can see a gap of about 140% between the end-of-year v1 calibration and the October 14' averaged IV calibration for portfolio 1. This is a clear example of the locality in space and time of market-consistency.

[2]See the paper for more information about the portfolios tested.

[3]The economic own funds valuation presents the same issues as for the economic liabilities Best Estimate as the market value of the Asset part in the economic balance sheet is obvious.

Table 1: Comparison between the obtained economic own funds values - 2014 results

	October 14’	Oct. & Nov. 14’	12/31/14 v1	12/31/14 v2
portfolio 1	16,898	15,614	7,046	10,000
portfolio 2	12,826	12,283	9,517	10,000
portfolio 3	12,553	12,073	6,050	10,000

2 Insurance market-consistency and the no-arbitrage condition

The El Karoui et al. (2017) proposition to improve the stability of market data used for insurance interest rates models calibration is often questioned by practitioners, in particular due to its non-respect of the no-arbitrage opportunity condition. Indeed, the averaged volatility matrices do not fit the yield curve date (end-of-year) which *a priori* leads to market prices arbitrage opportunities. In this Section we have chosen to demystify the no-arbitrage opportunity assumption, in particular as soon as the life insurance economic valuation is considered. After a short recall of the theoretical basics of this assumption, we focus on its various implications on financial market practices to better underline its non-adaptability to the insurance market. This leads us to show the limits of some regulation requirements and finally to refute the legitimacy of this counter-argument to the El Karoui et al. proposition.

2.1 Short reminder on risk-neutral theory and the no-arbitrage condition

Here we quickly recall the historical developments and underlying assumptions below the no-arbitrage principle.

Arrow and Debreu (see Debreu (1987), Arrow (1964) and Arrow & Debreu (1954)) determine the optimal probability measure to balance prices in dynamic arbitrage-free markets. Their work offered a solution to the set of supply/demand equations from the seminal Walras (1896) which first led to the idea of optimal price on an economy where all supply and demand act rationally, under perfect competition.

In parallel to the economic theory, the mathematical finance theory evolved from Bachelier (1900), to shape the conditions for the existence and unicity of a risk-neutral probability. It exists under the no-arbitrage condition (it is an equivalence) and is unique if the market is complete (all derivatives can be replicated by a linear strategy of simple assets).

Under these simple axioms, Black & Scholes (1973) and Merton (1973) develop risk-neutral models to formalise the stocks evolution (Geometric Brownian) and derivative prices at all time. Then many different models are developed for all sorts of financial assets (see Brigo & Mercurio (2006), Cont (2004),...).

The no-arbitrage condition induces the non-existence of any 0-cost market strategy that ensures a strictly positive gain. Nobel price Milton Friedman talked of *no free lunch*. Under the complete market assumption, the risk-neutral measure is the only valuation probability measure that ensures the no-arbitrage condition.

In practice, it is highly important to keep in mind that any use of a risk-neutral model for pricing (*e.g.* a derivative) assumes the valuated asset is well hedged (through market completeness) and only supports a risk purely associated to its underlyings and not to the general market. This is why the risk-neutral models only focus on underlying's volatilities and assume a risk-free averaged evolution of all assets. Out of this scheme the risk-neutral valuation has no reality. It is (for example) not used in corporate finance and for banking investment strategies, but only for derivative pricing. [4]

2.2 The no-arbitrage condition in financial markets vs. in Solvency II

2.2.1 A financial market prerequisite

In practice every financial market presents limited arbitrage opportunities at all times. These are limited and quickly cancelled as soon as the market is liquid enough. Indeed, if enough buyers and sellers are active, well-informed and logical, perfect competition is close and, as the number of financial agreements rises, the situation converges towards fair prices and arbitrage reduction.

This is an essential point to ensure the confidence of investors and of the market. On the 23 April 2013, after a fake tweet of Associated Press announcing two explosions in the White House, a quick arbitrage opportunity occurred in the Dow Jones. The index decreases by 1% but two minutes later recovers to its initial level (see Figure 1).

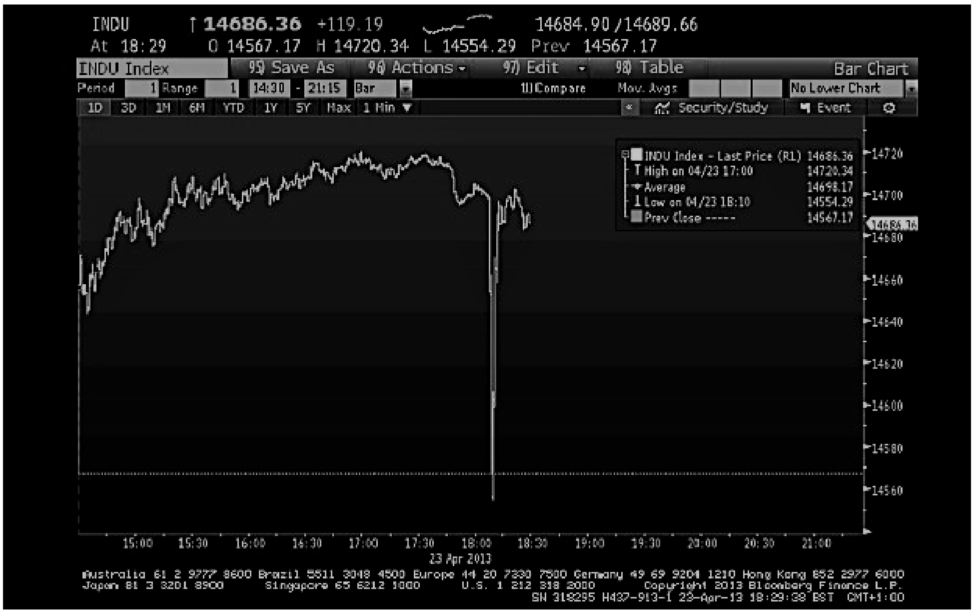

Figure 1: Arbitrage Opportunity in the Dow Jones - 04/23/13 (source : The Telegraph)

Such an issue in the transparency of information barely happens, in particular on such a liquid market. And the fast recovery, almost exact, eases investors concerning the index fair price. The no-arbitrage condition is an absolute prerequisite to efficient market deals and financial markets well-being.

[4]This is one of the major theoretical issue for its use in life insurance liabilities valuation, tools that are simply neither sold on complete markets nor hedgeable / hedged.

2.2.2 An insurance regulator requirement

From the Official Journal of the European Union volume 58 (2015) supplementing directive Solvency II, some straightforward notification

SECTION 3.1 - Methodologies to calculate technical provisions - Assumptions underlying the calculation of technical provisions - Article 22 (General provisions) paragraph 3 (European Commission (2015*a*)),

> Insurance and reinsurance undertakings shall set assumptions on future financial market parameters or scenarios that are appropriate and consistent with Article 75 of Directive 2009/138/EC. Where insurance and reinsurance undertakings use a model to produce projections of future financial market parameters, it shall comply with all of the following requirements:
>
> a it generates asset prices that are consistent with asset prices observed in financial markets;
>
> b it assumes no-arbitrage opportunity;
>
> c the calibration of the parameters and scenarios is consistent with the relevant risk-free interest rate term structure used to calculate the best estimate as referred to in Article 77(2) of Directive 2009/138/EC. [Solvency II]

The consistency with the relevant risk-free interest rate term structure for both calibration and simulation is fixed and verified in practice but the consistency with market prices is not clear, apart when considering that in fact the texts means IV. However, the second condition needs further discussions at this stage of our presentation.

2.3 The no-arbitrage condition implications for the insurance market

As soon as a different yield curve but similar market IV are considered, new option prices are deduced and there exist arbitrage opportunities with financial markets, see Figure 2 below, for such differences in swaption prices as of end-of-year 2017.

At first, one could assume the insurance regulator does not mean comparing finance and insurance markets prices in its view of ”no-arbitrage opportunity” in practice. Another possibility is that it only considers no-arbitrage on the (at least European) insurance market. But in practice there are still many arbitrages, such as for example, the swaption prices assessed using the European companies economic scenarios tables. Such options are used to calibrate the stochastic interest rates models which should lead to similar prices when simulating, but the maturities/tenors used to calibrate are not fixed by the regulator so that there exist many discrepancies among companies’ choices even in the same country. In addition, different models and different calibration algorithms are used, leading to different market option prices and therefore arbitrage opportunities directly embedded in the various companies economic scenarios tables.

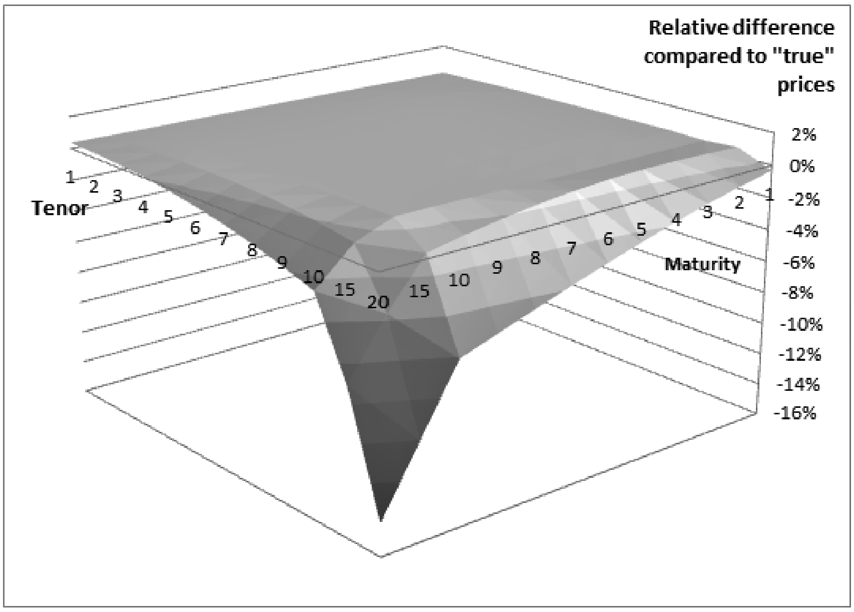

Figure 2: EIOPA swaption prices with the regulatory RFR yield curve vs. true prices (using the Euroswap yield curve) - matrix 20y×20y as at end-of-year 2017

In fact the only possibility to apply comparability of prices on the insurance economic values market would be to force the companies to use the same economic scenarios table.

Even if it were possible, actuaries are dealing with a market on which no one buys or sells products. In this context, prices obtained are never submitted to the supply and demand criteria and, unlike liquid financial markets, cannot converge towards fair prices.

Finally it must be admitted that the no-arbitrage condition has not the same implications here as for financial markets and practices. The "no-arbitrage opportunity" condition only states a valuation scheme where some market data and regulation choices are mixed to display more or less homogeneously obtained values. The fact this condition cannot be implemented for the insurance market should not be used as an excuse to legitimize insurance sector's practices.

These conclusions do not invalidate the insurance valuation scheme entirely, but aim at partly disconnecting its current implementation from the theoretical and practical financial market. This last framework is very complex and must be wisely balanced and understood when used as a theoretical and practical justification for insurance solvency regulation choices.

3 Economic valuation stability through estimators convergence acceleration

Every economic valuation requires economic scenarios tables (see El Karoui et al. (2017) for more insights about economic valuation methodology and Vedani & Devineau (2013) for its practical standard and approximated implementation approaches). The practical methods to assess all type of values (be it asset, liability best estimate or own funds) are very similar. For this reason we focus our discussion on economic own funds (or Net Asset Values) and more precisely on the Value of In-Force (VIF), that is the part of own funds ($OF = VIF + RNA$ [the Revalued Net Asset]) obtained through Monte-Carlo estimation of the averaged Net Present Value of future margins cash-flows (denoted by NPV below).

3.1 Monte-Carlo estimators convergence - a general issue in the Solvency II directive

One economic scenario leads to one valuation of NPV. If one considers an economic scenarios table of S scenarios one obtains a Monte-Carlo estimator of the VIF through,

$\widehat{VIF} = \frac{1}{S}\sum_{s=1}^{S} NPV^s.$

The Monte Carlo estimators convergence results from the Law of Large Numbers, without any additional model specification assumption. But the main issue with every Monte-Carlo valuation approach is its time complexity. The higher the variance of the underlying variable (NPV here), the higher the number of scenarios needed to provide an efficient estimator convergence.

VIF estimation therefore requires a high number of economic scenarios and corresponding NPV assessment (because the NPV variable has in general a very high variance). This is easily seen through the $0 < p < 100\%$ asymptotic confidence interval that can be estimated for the $\widehat{VIF}$ estimator,

$$CI_p\left(\widehat{VIF}\right) = \left[\widehat{VIF} \pm \frac{\sqrt{\hat{\mathbb{V}}[NPV]}\, q_{\frac{1+p}{2}}[\mathcal{N}(0,1)]}{\sqrt{S}}\right],$$

with the usual unbiased variance estimator,

$$\hat{\mathbb{V}}[NPV] = \frac{1}{(S-1)}\sum_{s=1}^{S}\left(NPV^s - \widehat{VIF}\right)^2.$$

To increase the estimator convergence speed (and decrease the confidence interval length), it is possible to increase S but it is also possible to try to decrease the variance of the variable of interest, NPV here. We present two possibilities to do so in subsection 3.2.

In reality two different estimators are of great importance concerning the Solvency II regulation. Along with the VIF estimator, the Solvency Capital Requirement (SCR) denotes the regulatory capital. According to the regulation the latter must be assessed at least every end-of-year and it can be calculated through two different approaches, Internal Model or Standard Formula (see in particular Devineau & Loisel (2009) which depicts the way both are theoretically implemented and the assumptions under which both approaches converge). In this paper we focus on the standard formula approach, which is the most widespread estimation method used.

This approach relies on the estimation of marginal risk capitals (stock risk, interest rates risk, longevity risk, lapse risk etc.), assessed as the difference between the standard (central) economic own funds and the marginally (instantaneously) shocked own funds. Assuming there exist I different shocks, any marginal SCR (denoted by $SCR_i, \forall i \in (risk_1, ..., risk_I)$) is obtained as,

$SCR_i = OF - OF_i = VIF - VIF_i$ with OF_i (respectively VIF_i) the i-marginally shocked own fund (respectively VIF), estimated by,

$\widehat{SCR}_i = \widehat{VIF} - \widehat{VIF}_i$.

These estimated sub-capitals are then aggregated through an elliptic approach in order to assess $\widehat{SCR}$, the estimated standard formula SCR.[5]

In subsection 3.3 we address the estimator $\widehat{SCR}_i$ convergence issue and focus on a specific methodological aspect, important to respect, that enables a quicker convergence of the estimated sub-capitals and finally of the estimated standard formula SCR (see Appendix 1).

3.2 VIF estimator convergence - variance reduction methods

Two different easy techniques can be used to decrease the NPV variance and increase the VIF estimator convergence. One of these approaches is often well-known and used by actuarial practitioners, the antithetic variables. But the second one is as easy to implement, generally very efficient, though not used by practitioners, namely control variate. It is very important to note that these two techniques can be implemented together to improve the convergence even more.

3.2.1 Antithetic variables

The antithetic variables (AV) methodology is a well-known variance reduction approach, often used in finance and simulation theory in general (see *e.g.* Paskov & Traub (1995) or Glasserman (2004)).

The idea behind most variance reduction techniques as is the case here is to speed up the Monte Carlo estimation of the interest variable average by considering an alternative variable with same mean but lower variance.

In practice each economic scenario can be summarised by a set of random standard Gaussian outcomes, those used to simulate its underlying economic variables (stock indexes, yield curves, spreads,...). Let ε be this sample of outcomes so that one can see any valuated NPV as a closed function of ε, which we simply note below (when necessary) $NPV(\varepsilon)$.

One finally estimates, with evident notations (using $NPV(\varepsilon^s)$ instead of NPV^s),

$\widehat{VIF} = \frac{1}{S}\sum_{s=1}^{S} NPV(\varepsilon^s)$.

The application of antithetic variables to VIF estimation consists in considering S even ($S = 2N$ below), and, denoting by $(\varepsilon^s)_{s\in[\![1;N]\!]}$ the N first scenarios random outcomes vectors, in using $\left(\varepsilon^s = -\varepsilon^{s-N}\right)$ $\forall s \in [\![N+1;S]\!]$ for the N last scenarios random outcomes vectors.

In practice, all conditions sum up to $CoVar[NPV(\varepsilon);NPV(-\varepsilon)] \leq 0$ be always verified whichever asset-liability management (ALM) model or economic conditions are considered. This leads to,

$\mathbb{V}\left[\widehat{VIF}^{AV}\right] = \mathbb{V}\left[\frac{1}{S}\sum_{s=1}^{S} NPV(\varepsilon^s)\right]$

$= \mathbb{V}\left[\frac{1}{S}\sum_{s=1}^{N} NPV(\varepsilon^s) + NPV(-\varepsilon^s)\right]$

[5] In reality the *full aggregated SCR* is a bit more complex but we chose to focus on an elliptic aggregation for the sake of simplicity.

$= \frac{1}{2S}\mathbb{V}\left[NPV(\varepsilon)+NPV(-\varepsilon)\right]$

$= \frac{1}{2S}\left(\mathbb{V}\left[NPV(\varepsilon)\right]+\mathbb{V}[NPV(-\varepsilon)]+2CoVar\left[NPV(\varepsilon);NPV(-\varepsilon)\right]\right)$

$\leq \frac{1}{2S}\left(\mathbb{V}\left[NPV(\varepsilon)\right]+\mathbb{V}\left[NPV(-\varepsilon)\right]\right),$

and, as $\mathbb{V}\left[NPV(\varepsilon)\right]=\mathbb{V}\left[NPV(-\varepsilon)\right]$(recall ε is a standard Gaussian outcomes vector),

$\mathbb{V}\left[\widehat{VIF}^{AV}\right] \leq \frac{1}{S}\left(\mathbb{V}\left[NPV(\varepsilon)\right]\right) = \mathbb{V}\left[\frac{1}{S}\sum_{s=1}^{S} NPV(\varepsilon^s)\right] = \mathbb{V}\left[\widehat{VIF}\right].$

Finally the variance of the AV-VIF estimator is lower than the one of a standard VIF estimator, that would have been estimated using S independent random outcomes vectors (S independent economic scenarios). This justifies the use of antithetic variables as a VIF estimator variance reduction. The AV estimator converges faster (and, in particular, the confidence interval length decreases).

3.2.2 Control variate

Control variate (CV) is another variance reduction methodology, also very well-known in finance and simulation theory (see *e.g.* Hull & White (1988) or again Glasserman (2004)), that can be easily adapted to our methodology.

The underlying idea of CV is to find a variable Y, also function of ε, whose exact average is known and which satisfies $CoVar\left[NPV(\varepsilon);Y(\varepsilon)\right] \neq 0$.

In practice such variables are easy to find for insurance NPV. The NPV are highly linked to the movements of the economic variables embedded within the economic scenarios their calculation relies on. Finally, one can select a portfolio of assets which cash-flows are highly correlated to the net present values of margin cash-flows. If these assets are not too complex, their true expectancy at $t = 0$ (their prices) is straightforward and finally the portfolio true averaged value at $t = 0$ is known. In this example Y is the portfolio net present value of future cash flows, which fully depends on the considered economic scenarios and therefore on its ε vector. This type of portfolio variable, highly correlated with the insurance NPV has already been investigated and is the base of the replicating portfolio theory (see *e.g.* Schrager (2008), Devineau & Chauvigny (2011), Pelsser & Schweizer (2016)).

The implementation of CV still relies on the use of an alternative variable of interest. Instead of $NPV(\varepsilon)$, the practitioner may use a variable with the same average but a lower variance, the CV-$NPV(\varepsilon)$, denoted by $NPV^{CV}(\varepsilon)$ and defined as,

$NPV^{CV}(\varepsilon) = NPV(\varepsilon) + c\left(Y(\varepsilon) - \mathbb{E}\left[Y(\varepsilon)\right]\right),$

with c a constant. One verifies,

$$\begin{cases} \mathbb{E}\left[NPV(\varepsilon)+c\left(Y(\varepsilon)-\mathbb{E}\left[Y(\varepsilon)\right]\right)\right] = \mathbb{E}\left[NPV(\varepsilon)\right] = VIF, \text{ and,} \\ \mathbb{V}\left[NPV(\varepsilon)+c\left(Y(\varepsilon)-\mathbb{E}\left[Y(\varepsilon)\right]\right)\right] = \mathbb{V}\left[NPV(\varepsilon)\right] + c^2\mathbb{V}\left[Y(\varepsilon)\right] + 2c.CoVar\left[NPV(\varepsilon);Y(\varepsilon)\right]. \end{cases}$$

And if c is chosen to minimise the variance, that is, $c^* = -\frac{CoVar[NPV(\varepsilon);Y(\varepsilon)]}{\mathbb{V}[Y(\varepsilon)]}$ one verifies that,

$\mathbb{V}\left[NPV(\varepsilon)+c^*\left(Y(\varepsilon)-\mathbb{E}\left[Y(\varepsilon)\right]\right)\right] \leq \mathbb{V}\left[NPV(\varepsilon)\right]$ (obtained for $c = 0$).

Finally we obtain,

$\mathbb{V}\left[\widehat{VIF}^{CV}\right] = \mathbb{V}\left[\frac{1}{S}\sum_{s=1}^{S} NPV^{CV}(\varepsilon^s)\right] = \frac{1}{S}\mathbb{V}\left[NPV^{CV}(\varepsilon)\right] \leq \frac{1}{S}\mathbb{V}\left[NPV(\varepsilon)\right] = \mathbb{V}\left[\widehat{VIF}\right]$.

This justifies the use of control variate as a VIF estimator variance reduction. The CV estimator converges faster (and the confidence interval length decreases).

As a matter of fact, if $CoVar\left[NPV(\varepsilon); Y(\varepsilon)\right] = 0$ then $c^* = 0$ and $NPV(\varepsilon^s)^{CV} = NPV(\varepsilon^s)$, the CV has no impact. In opposition, if $NPV(\varepsilon)$ and $Y(\varepsilon)$ are collinear/antilinear, then $\mathbb{E}\left[\widehat{VIF}\right]$ is known up to a multiplicative coefficient $\left(\mathbb{E}\left[\widehat{VIF}\right] = \pm\sqrt{\frac{\mathbb{V}[NPV(\varepsilon)]}{\mathbb{V}[Y(\varepsilon)]}}\mathbb{E}\left[Y(\varepsilon)\right]\right)$.

Besides, in all generality $Correl\left[NPV(\varepsilon); Y(\varepsilon)\right] \in]-1;0[\cup]0;1[$, and CV users aim to get Y such as this correlation is the closest to -1 or 1.

Once again, note that AV and CV can totally be implemented at the same time without full cannibalisation of their positive effects. The final AV+CV VIF estimator converges even faster.

3.3 Standard Formula *SCR* convergence - the simulation seed importance

In this subsection we address the standard formula *SCR* estimator convergence issue. We focus here on one single sub-risk *SCR* estimator but the point raised can be applied to all marginal *SCR* estimator, so that the final *SCR* estimator variance should decrease (see Appendix 1).

Recall the shape of our marginal *SCR* estimator associated to shock i,

$\widehat{SCR}_i = \widehat{VIF} - \widehat{VIF}_i$.

Under the notation of subsection 3.2 we can also express the estimator as,

$\widehat{SCR}_i = \frac{1}{S}\sum_{s=1}^{S} NPV(\varepsilon^s) - \frac{1}{S}\sum_{s=1}^{S} NPV_i(\varepsilon^{n_s})$, with $n_s \in \mathbb{N}$ for all $s \in [\![1;S]\!]$, or,

$\widehat{SCR}_i = \frac{1}{S}\sum_{s=1}^{S}\left(NPV(\varepsilon^s) - NPV_i(\varepsilon^{n_s})\right)$.

In practice we want to underline here the importance to conserve the same simulation seed in both the economic scenarios used to asses the standard VIF and those used to assess the shocked VIF. One therefore has (denoting by $^{s.s}$ when using the same simulation seed in our estimations, be it of OF, VIF, full standard formula or marginal *SCR*),

$\widehat{SCR}_i^{s.s} = \frac{1}{S}\sum_{s=1}^{S}\left(NPV(\varepsilon^s) - NPV_i(\varepsilon^s)\right)$.

This point is of great importance because one generally verify that $CoVar\left[NPV(\varepsilon); NPV_i(\varepsilon)\right] > 0$ so that,

$\mathbb{V}\left[\widehat{SCR}_i^{s.s}\right] = \frac{1}{S}\mathbb{V}\left[NPV(\varepsilon) - NPV_i(\varepsilon)\right] = \frac{1}{S}\left(\mathbb{V}\left[NPV(\varepsilon)\right] + \mathbb{V}\left[NPV_i(\varepsilon)\right]\right) - \frac{2}{S}CoVar[NPV(\varepsilon); NPV_i(\varepsilon)]$

$\leq \frac{1}{S}\left(\mathbb{V}\left[NPV(\varepsilon)\right] + \mathbb{V}\left[NPV_i(\varepsilon)\right]\right) = \mathbb{V}\left[\widehat{SCR}_i\right]$.

This upper bound is generally satisfied in practice. We have here tested this point on a standard life insurance savings product, on a marginally shocked *SCR*, based on 2,500 economic scenarios, with results displayed in Table 2 (denoting by ρ the empirical correlation between the central and shocked NPV),

Table 2: Comparison between the approximated marginal *SCR* values - same seed vs. different seeds with 95% confidence intervals

***Spread** SCR*	**Central** *VIF*	**Shocked** *VIF*	**Marginal** *SCR*	ρ
Different seeds	11,289 ($\pm$ 369)	-441 ($\pm$ 408)	11,730 ($\pm$ 553)	$\rho = -0.7\%$
Same seed	11,289 ($\pm$ 369)	-448 ($\pm$ 416)	11,737 ($\pm$ 142)	$\rho = 94.2\%$
***Stock** SCR*	**Central** *VIF*	**Shocked** *VIF*	**Marginal** *SCR*	ρ
Different seeds	11,289 ($\pm$ 369)	3,789 ($\pm$ 377)	7,500 ($\pm$ 525)	$\rho = 1.0\%$
Same seed	11,289 ($\pm$ 369)	3,831 ($\pm$ 382)	7,458 ($\pm$ 104)	$\rho = 96.3\%$
***Real estate** SCR*	**Central** *VIF*	**Shocked** *VIF*	**Marginal** *SCR*	ρ
Different seed	11,289 ($\pm$ 369)	8,724 ($\pm$ 512)	2,565 ($\pm$ 512)	$\rho = 1.4\%$
Same seed	11,289 ($\pm$ 369)	8,618 ($\pm$ 376)	2,671 ($\pm$ 54)	$\rho = 98.9\%$

The results show a direct application of the previous inequality. Whichever the shocked *SCR*, the *Different seeds* estimated *SCR* converges much slower than the *Same seed* estimated *SCR* (not the estimated VIF whose confidence intervals don't evolve). In particular, concerning the spread *SCR*, we obtain a decrease of the half-confidence interval by a coefficient $\frac{553}{142} \approx 3.9$. If a different seed had been used, this means the number of scenarios that should have been used to evaluate the marginal *SCR* with the same efficiency would have been close to $2500 \times 3.9^2 = 38025$.

This empirically justifies the use of the same seed, when estimating standard and shocked VIF, as a marginal *SCR* estimator variance reduction. Finally we assume the elliptically aggregated *SCR* convergence increases (we verify it empirically in Appendix 1).

4 Implementation - a control variate specific application to increase the VIF convergence speed

In this Section we present a relatively easy application of the methods from Section 3 using both variance reduction methods presented and the seed conservation impact to increase even more the VIF estimators convergence speed.

4.1 Implementation context and formalisation

All our work has been ignited by the empirical analysis of various ALM models. From an simplistic point of view, ALM models are complex accumulations of accounting functions with entity specific parameters (contract parameters, loading factors,...) and inputs (assets, liability model points,...). Both are regularly updated but embed a strong inertia due to various factors including long-term contracts, top-management choices and long-term objectives,...)

Finally the way ALM models behave to an economic scenario random vector ε stays rather similar, whichever initial economic conditions and calibration data are chosen by the modelling team. At least on a limited time-lapse.

Still using the seed-keeping character, leading to correlated central NPV and shocked NPV_i, we have tried to estimate the correlation of $NPV_\alpha(\varepsilon)$ and of $NPV_\beta(\varepsilon)$, with NPV_α and NPV_β two different NPV variables associated to two different economic conditions (namely situations α and β).

The proposed methodology is simply to use NPV_α (and the true value of $\mathbb{E}[NPV_\alpha]$, denoted by VIF_α below), as a control variate for any other VIF one wants to estimate, here denoted as VIF_β (estimated on NPV_β outcomes).

The main issue here is to assess an efficient estimation of VIF_α. This is why we only implement it one time though we test it on several different VIF_β economic situations.

In the implementations below we have chosen to estimate this VIF_α value using 50,000 economic scenarios so that the convergence of our estimator is good enough to be very close to the true VIF_α. We abusively identify this estimator to the true VIF_α below (see subsection 4.3 for a more precise development on the implications of the imperfect equality between this estimator and the true VIF_α value).

This VIF_α is estimated based on the end-of-year 2017 regulatory-consistent economic conditions for all simulated economic quantity but nominal rates. Concerning nominal rates, we have used as the reference (calibration/simulation) yield curve the market Euroswap rates (not the regulatory EIOPA yield curve, used in Implementation 1 for the VIF_β situation), and the calibration swaption IV matrix is the averaged 10 (maturities) x 10 (tenors) December 2017 matrix (not the *standard* end-of-year matrix, also used in Implementation 1 for the VIF_β situation). This allows us to test very different economic conditions for VIF_β. We assume any other economic situation chosen as our reference VIF_α's should have led to similar results[6], we have simply chosen one in all generality.

We obtain for 50,000 scenarios, using AV, and with a randomly chosen simulation seed,[7]

$VIF_\alpha = \mathbb{E}[NPV_\alpha] \approx \widehat{\mathbb{E}}[NPV_\alpha] = 22{,}005 \pm 765.$

4.2 Implementation of control variate using VIF_α

Recall subsection 3.2.2 concerning control variate implementation. We want to improve the convergence of an estimator $\widehat{VIF}_\beta = \frac{1}{S}\sum_{s=1}^{S} NPV_\beta(\varepsilon^s)$ using a control variate, here NPV_α outcomes, assumed well correlated with the $NPV_\beta(\varepsilon^s)_{s\in[\![1;S]\!]}$, and knowing $\mathbb{E}[NPV_\alpha] = VIF_\alpha$ (see 4.1 for more details).

Indeed, if we already know VIF_α quite well, we also need NPV_α and NPV_β correlated (the most possible) outcomes. In the implementations below we therefore always consider the same simulation seed to search for a strong correlation between our NPV_α and NPV_β outcomes. Basically this means we have to assess couples $\left(NPV_\alpha(\varepsilon^s), NPV_\beta(\varepsilon^s)\right)_{s\in[\![1;S]\!]}$. In addition we want to stay operationally realistic, with S low enough to be implementable. We have therefore chosen below $S = 2{,}000$ scenarios using antithetic variables.

We obtain for 2,000 scenarios, using AV, and with our fixed simulation seed,

[6] The estimation actually depends on the choice of the reference date but the numerical evidence displayed in the article are here to illustrate the effect without any objective of exhaustivity rather than a thorough study of the reference date choice.

[7] Note that in this section we focus on recent yield curves including negative interest rates. For this reason we use a Displaced Diffusion Libor Market Model (see in particular Joshi & Rebonato (2003)) to model interest rates and calibrate these models on normal/Bachelier swaptions IV (see *e.g.* Hohmann et al. (2015)). These choices are realistic considering current actuarial practices.

$\widehat{VIF}_\alpha = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000} NPV_\alpha(\varepsilon^s) = 23{,}949 \pm 3{,}666.$

4.2.1 Implementation 1: VIF_α economic situation vs. the *standard* regulatory economic conditions as of end-of year 2017

In this first implementation we use the full standard end-of-year 2017 regulatory-consistent conditions for the VIF_β economic situation. We therefore try to assess the true regulatory VIF of our savings product as of end-of-year 2017. Compared to our VIF_α situation, we therefore change the calibration swaption IV matrix, using the end-of-year 10x10 matrix, and we use the corresponding RFR to calibrate and simulate the 2,000 economic scenarios associated with the $(\varepsilon^s)_{s\in[\![1;2{,}000]\!]}$.

We obtain,

$\widehat{VIF}_\beta = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000} NPV_\beta(\varepsilon^s) = 20{,}615 \pm 3904.$

As expected, based on the couples $\left(NPV_\alpha(\varepsilon^s), NPV_\beta(\varepsilon^s)\right)_{s\in[\![1;2{,}000]\!]}$, we estimate,

$\widehat{Correl}[NPV_\alpha(\varepsilon), NPV_\beta(\varepsilon)] = 96.9\%.$

We have also,

$\hat{c}^* = -\frac{\widehat{CoVar}\left[NPV_\alpha(\varepsilon);NPV_\beta(\varepsilon)\right]}{\hat{\mathbb{V}}[NPV_\alpha(\varepsilon)]} = -1.03.$

And eventually,

$\widehat{VIF}_\beta^{CV} = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000}\left(NPV_\beta(\varepsilon^s) + \hat{c}^*\left(NPV_\alpha(\varepsilon^s) - VIF_\alpha\right)\right) = 18{,}609 \pm 961.$

We obtain a decrease of the half-confidence interval by a coefficient $\frac{3{,}904}{961} \approx 4.1$. This means without using the CV such efficient estimation would have required close to $2{,}000 \times 4.1^2 = 33{,}620$ scenarios.

4.2.2 Implementation 2: VIF_α economic situation vs. an updated *El Karoui et al. (2017)* calibration approach

In this second implementation we consider for VIF_β the situation tested and favoured in El Karoui et al. (2017) but adapted to end-of-year 2017. Compared to our VIF_α situation, we change the calibration swaption IV matrix and use the averaged October-November 2017 *v2* matrix (see subsection 1.3), and we use the RFR to calibrate and simulate the 2,000 economic scenarios associated with the $(\varepsilon^s)_{s\in[\![1;2{,}000]\!]}$.

We obtain,

$\widehat{VIF}_\beta = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000} NPV_\beta(\varepsilon^s) = 11{,}354 \pm 4{,}336.$

As expected, based on the couples $\left(NPV_\alpha(\varepsilon^s), NPV_\beta(\varepsilon^s)\right)_{s\in[\![1;2{,}000]\!]}$, we estimate,

$\widehat{Correl}[NPV_\alpha(\varepsilon), NPV_\beta(\varepsilon)] = 92.9\%.$

We have also,

$\hat{c}^* = -\frac{\widehat{CoVar}\left[NPV_\alpha(\varepsilon);NPV_\beta(\varepsilon)\right]}{\hat{\mathbb{V}}[NPV_\alpha(\varepsilon)]} = -1.10.$

And eventually,

$\widehat{VIF}_\beta^{CV} = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000}\left(NPV_\beta(\varepsilon^s) + \hat{c}^*\left(NPV_\alpha(\varepsilon^s) - VIF_\alpha\right)\right) = 9{,}219 \pm 1{,}602.$

We obtain a decrease of the half-confidence interval by a coefficient $\frac{4{,}336}{1{,}602} \approx 2.7$. This means without using the CV such efficient estimation would have required close to $2{,}000 \times 2.7^2 = 14{,}580$ scenarios.

4.2.3 Implementation 3: VIF_α economic situation vs. the regulatory conditions as of end of June (mid year) 2018

In this third implementation we use the full standard end of June 2018 regulatory conditions for the VIF_β economic situation. We therefore try to assess the true regulatory VIF of our savings product as of end of June 2018. Compared to our VIF_α situation, we therefore change the calibration swaption IV matrix, using the end of June 2018 10x10 matrix, and we use the corresponding RFR to calibrate and simulate the 2,000 economic scenarios associated with the $(\varepsilon^s)_{s\in[\![1;2{,}000]\!]}$.

We obtain,

$\widehat{VIF}_\beta = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000} NPV_\beta(\varepsilon^s) = 38{,}470 \pm 3{,}667.$

As expected, based on the couples $\left(NPV_\alpha(\varepsilon^s), NPV_\beta(\varepsilon^s)\right)_{s\in[\![1;2{,}000]\!]}$, we estimate,

$\widehat{Correl}[NPV_\alpha(\varepsilon), NPV_\beta(\varepsilon)] = 91.9\%.$

We have also,

$\hat{c}^* = -\frac{\widehat{CoVar}\left[NPV_\alpha(\varepsilon); NPV_\beta(\varepsilon)\right]}{\hat{\mathbb{V}}[NPV_\alpha(\varepsilon)]} = -0.92.$

And eventually,

$\widehat{VIF}_\beta^{CV} = \frac{1}{2{,}000}\sum_{s=1}^{2{,}000}\left(NPV_\beta(\varepsilon^s) + \hat{c}^*\left(NPV_\alpha(\varepsilon^s) - VIF_\alpha\right)\right) = 36{,}682 \pm 1{,}447.$

We obtain a decrease of the half-confidence interval by a coefficient $\frac{3{,}667}{1{,}447} \approx 2.5$. This means without using the CV such efficient estimation would have required close to $2{,}000 \times 2.5^2 = 12{,}500$ scenarios.

4.3 Final analyse based on our implementations results

The methodology presented and implemented in Section 4 may seem practically delicate to use but, as aforementioned, its unique complexity lies in the very precise estimation of one VIF (denoted by VIF_α above, estimated by 50,000 scenarios here)[8]. Then, using control variate enables to take profit of the convergence knowledge assessed from this heavy estimation and improve any other VIF_β valuation, as soon as the economic situation under which this valuation is made is not too far (in time and in space) from the VIF_α situation.

Through our different implementations we have obtained, with our implementation choices, an im-

[8] Concerning the NPV_α outcomes required to build the control variate part of the NPV_β^{CV} ($\hat{c}^*\left(NPV_\alpha(\varepsilon^s) - \widehat{VIF}_\alpha\right)$) it seems also possible to use the same seed when estimating precisely VIF_α as when estimating roughly (without using CV) VIF_β so that the NPV_α outcomes are already calculated when estimating the precise VIF_α estimation and they do not need additional complexity.

provement of the estimators variance reduction equivalent to using between 6 and 20× more economic scenarios. However one must note that this great improvement is also associated to the integration of a relatively small bias to the valuation linked to the fact that the true VIF_α is not known in reality, we have simply an (good) estimation $\widetilde{VIF}_\alpha = \frac{1}{50{,}000}\sum_{s=1}^{50{,}000} NPV_\alpha(\varepsilon^{v_s})$ for some $v_s \in \mathbb{N}$, $\forall s \in [\![1; 50{,}000]\!]$.[9] Finally,

$$\mathbb{E}\left[\widehat{VIF}_\beta^{CV}\right] = \mathbb{E}\left[NPV_\beta^{CV}(\varepsilon)\right] = \mathbb{E}\left[NPV_\beta(\varepsilon) + \hat{c}^*\left(NPV_\alpha(\varepsilon) - \widetilde{VIF}_\alpha\right)\right]$$
$$= VIF_\beta + \hat{c}^*\left(VIF_\alpha - \widetilde{VIF}_\alpha\right).$$

The final bias of $\widehat{VIF}_\beta^{CV}$ is therefore $\hat{c}^*\left(VIF_\alpha - \widetilde{VIF}_\alpha\right)$. This bias is small and would be null if the estimator $\widetilde{VIF}_\alpha$ had converged. To reduce this bias it is therefore ideal to use the highest possible number of economic scenarios when estimating the *true* VIF_α value. We consider the convergence improvement as relevant enough to prevail this small bias that can be controlled.

Basically, the underlying idea of this methodology is to obtain a footprint of the ALM model and its behaviour, at one moment in time. Then using this knowledge, the user knows how to adjust future ALM valuations. Under this comparison, it seems *a priori* necessary to keep the same ALM model. This is indeed ideal but in practice most ALM models behave similarly to similar economic scenarios. This means any two models would actually display grossly the same variations even if those are not exactly one-to-one. For example, when stock indexes rise the *NPV* rises, whichever ALM model is considered. It is impossible to generalise it to full scenarios that include stock indexes, interest rates (nominal/real/risky),... But we can assume a change in ALM model between the VIF_α and VIF_β may conserve $Correl(NPV_\alpha(\varepsilon), NPV_\beta(\varepsilon)) > 0$.

Finally, we recall this approach would only lead to efficient results when regularly updated. Once again, the ALM model footprint is highly dependent on its underlying model points, Asset-Liability transition parameters and on many other points. We highly recommend, for the sake of efficiency, to update it at least yearly.

Conclusion

In this paper we have focused on the necessity to stabilise the European life insurance regulation valuation scheme based on the so-called concept of market-consistency. We have underlined the insurance-specific interpretation of this concept which originally tends to link the scheme to market finance practices and theory.

In section 1, we motivated the construction of the whole Solvency II Directive in its objective of enforcing financial stability or harmonising accounting norms to increase both transparency and financial resilience and as a consequence consumer protection. However, by doing so, different elements such as the Risk Free Rate or items from the Long term guarantee package - *e.g.* the volatility adjustment - have material consequences on the levels of the interest rate curve and as outlined in El Karoui et al.

[9] It seems possible to use $v_s = s, \forall s \in [\![1; 50{,}000]\!]$ (same seed) in particular when the number of simulations (50,000 here) is high enough.

(2017) intrinsically contradicts the goal of market-consistency as far as the market prices and technical provisions best estimates are concerned.

In particular we have presented, through Section 2, some practical reasons why the no-arbitrage opportunity condition can never be respected concerning insurance valuation. This aspect is only theoretical and does not question the practical regulation implementation framework. We wanted here to further investigate the mathematical background, especially regarding the actual interest of the averaging approach introduced by El Karoui et al. (2017), in order to introduce stability in the calibration data used by practitioners, but also in reducing the manipulability of the valuation scheme.

In Sections 3 and 4 we have then addressed a second aspect of major importance in the stability of the considered valuation. We have proposed different approaches to improve the Solvency II Monte-Carlo estimators convergence by working on the simulation seeds and on the use of variance reduction techniques. Finally, in our implementations we have reduced the variance of our VIF estimators by more than $16\times$ (but integrating a small - manageable - bias in estimation, see subsection 4.2 and 4.3) and the marginal (standard formula) *SCR* by up to more than $15\times$ (in subsection 3.3). Note that these results have been obtained on a single ALM life insurance model and future investigations will be necessary to efficiently generalize this analysis.

Though we consider the simulation seed issue we must recall to practitioners the only question that must be addressed here is convergence. Indeed, though it is probable that for a given model there exist some seeds leading to estimations closer to their true value than others, it cannot be generalised that one specific seed is better than another. Therefore we do not address the seed choice problem but only promote the choice of the same simulation seeds for each calculations, to valuate standard formula *SCR*, use the control variate method presented in Section 4 but also to respect some stability in estimation bias and improve the comparability through time of economic valuations.

In future works we will test our methods on further ALM models and economic situations. We also intend to address deeper the similar simulation seed implications on valuation, propose a complete update of the El Karoui et al. (2017) approach and test further sensitivities on economic valuations. Would those techniques be used on an industrial scale with all robustness checks being made, this would highlight and enhance the feasibility of forward looking analyses such as ORSA (or other type of estimations from the Pillar 2) which are highly time-consuming and resource-demanding.

Appendix 1 - impact of using similar simulation seed on aggregated *SCR*

Let us now consider 2 different marginal *SCR*, for risks i and $j \in (risk_\alpha, ..., risk_I)$ and denote by $\rho_{i,j}$ the elliptic aggregation correlation (users often choose the regulatory specifications which provide these correlations, see in particular European Insurance and Occupational Pensions Authority (2010)) between SCR_i and SCR_j. We consider below,

$$SCR_{i,j} = \sqrt{SCR_i^2 + SCR_j^2 + 2\rho_{i,j} SCR_i SCR_j}.$$ [10]

We aim at comparing $\mathbb{V}\left[\widehat{SCR}_{i,j}^{s.s}\right]$ and the standard $\mathbb{V}\left[\widehat{SCR}_{i,j}\right]$.

[10] Basically the aggregated *SCR* is obtained by $SCR = \sqrt{\sum_{i,j \in (risk_1,...,risk_I)} \rho_{i,j} SCR_i SCR_j}$ with $\rho_{i,j} = 1$ when $i = j$.

$\mathbb{V}\left[\widehat{SCR}_{i,j}\right] = \mathbb{V}\left[\sqrt{\widehat{SCR}_i^2 + \widehat{SCR}_j^2 + 2\rho_{i,j}\widehat{SCR}_i\widehat{SCR}_j}\right]$
$= \mathbb{E}\left[\widehat{SCR}_i^2 + \widehat{SCR}_j^2 + 2\rho_{i,j}\widehat{SCR}_i\widehat{SCR}_j\right] - SCR_{i,j}^2$
$= \mathbb{V}\left[\widehat{SCR}_i\right] + \mathbb{V}\left[\widehat{SCR}_j\right] + 2\rho_{i,j}\mathbb{E}\left[\widehat{SCR}_i\widehat{SCR}_j\right] + SCR_i^2 + SCR_j^2 - SCR_{i,j}^2.$

We can easily assume, from the work above,

$$\begin{cases} \mathbb{V}\left[\widehat{SCR}_i^{s.s}\right] \leq \mathbb{V}\left[\widehat{SCR}_i\right], \\ \mathbb{V}\left[\widehat{SCR}_j^{s.s}\right] \leq \mathbb{V}\left[\widehat{SCR}_j\right]. \end{cases}$$

Concerning the $+2\rho_{i,j}\mathbb{E}\left[\widehat{SCR}_i\widehat{SCR}_j\right]$ part it is complex to estimate a general comparison with the optimised $+2\rho_{i,j}\mathbb{E}\left[\widehat{SCR}_i^{s.s}\widehat{SCR}_j^{s.s}\right]$. We can assume $\rho_{i,j} \geq 0$ (true for most regulatory correlation factors) but it is probable that the comparability of both expressions depends on risks i and j... If $\rho_{i,j} = 0$ it is clear that $\mathbb{V}\left[\widehat{SCR}_{i,j}^{s.s}\right] \leq \mathbb{V}\left[\widehat{SCR}_{i,j}\right]$. In any case we assume the efficiency of keeping the same simulation seed to reduce the variances should also lead to $\mathbb{V}\left[\widehat{SCR}_{i,j}^{s.s}\right] \leq \mathbb{V}\left[\widehat{SCR}_{i,j}\right]$.

We have tested several $\rho_{i,j} > 0$ values and several couples of risks i and j with following results obtained using bootstrap estimators assessed on 2,500 aggregated *SCR* outcomes, each one estimated on 2,500 standard bootstrapped - sampled with replacement (see *e.g.* Efron & Tibshirani (1997)) - outcomes among 2,500 i.i.d. outcomes $NVP - NPV_i$. Using standard bootstrap it is possible to assess specific confidence intervals: $CI_{\alpha\%}^{bootstrap}\left(\widehat{SCR}\right) = \left[2\hat{\mathbb{E}}\left[\widehat{SCR}\right] - q_{\frac{1+\alpha}{2}}\left(\widehat{SCR}\right); 2\hat{\mathbb{E}}\left[\widehat{SCR}\right] - q_{\frac{1-\alpha}{2}}\left(\widehat{SCR}\right)\right]$

The results are given in Table 3.

Table 3: Comparison between the approximated 2-risks aggregated *SCR* values - same seed vs. different seeds with 95% confidence interval

***Spread/stock** SCR*	$\rho_{i,j} = 0.25$	$\rho_{i,j} = 0.5$	$\rho_{i,j} = 0.75$
Different seeds	15,427 [14,482;16,032]	16,793 [15,766;17,460]	18,055 [16,936;18,778]
Same seed	15,398 [15,184;15,545]	16,759 [16,525;16,920]	18,017 [17,765;18,180]
***Spread/RE** SCR*	$\rho_{i,j} = 0.25$	$\rho_{i,j} = 0.5$	$\rho_{i,j} = 0.75$
Different seeds	12,617 [11,746;13,180]	13,198 [12,234;13,820]	13,755 [12,689;14,455]
Same seed	12,673 [12,472;12,799]	13,277 [13,066;13,411]	13,855 [13,635;13,997]
***RE/stock** SCR*	$\rho_{i,j} = 0.25$	$\rho_{i,j} = 0.5$	$\rho_{i,j} = 0.75$
Different seeds	8,507 [7,628;9,053]	9,053 [8,089;9,668]	9,567 [8,526;10,252]
Same seed	8,526 [8,364;8,625]	9,091 [8,910;9,198]	9,623 [9,438;9,736]

Finally the use of the same simulation seed seems to work very well on the 2-risks aggregated *SCR* but also on the 3-risks aggregated *SCR* we tested (see Table 4).

Table 4: Comparison between the approximated RE/stock/spread risks aggregated *SCR* values - same seed vs. different seeds with 95% confidence interval

***3-risks** SCR*	$\rho_{i,j} = \rho_{i,k} = \rho_{j,k} = 0.25$	$\rho_{i,j} = \rho_{i,k} = \rho_{j,k} = 0.5$	$\rho_{i,j} = \rho_{i,k} = \rho_{j,k} = 0.75$
Different seeds	16,400 [15,249;17,142]	18,371 [17,019;19,241]	20,150 [18,603;21,146]
Same seed	16,426 [16,168;16,582]	18,417 [18,126;18,591]	20,213 [19,891;20,405]

Appendix 2 - concerning solvency ratios

We have proposed above different approaches to both reduce the variance of the OF/VIF and the SCR. Now we also address the solvency ratio ($SR = \frac{OF}{SCR}$ or using our both approaches and implicit notations, $SR^{CV} = \frac{OF^{CV}}{SCR^{CV,s.s}}$) issue. This ratio is in many ways the most important quantity used by practitioners and control authorities to evaluate and compare solvency among companies.As aforementioned, the variance of the OF is basically the one of the VIF ($OF = RNA + VIF$ and the RNA is deterministic).

First, it appears natural to couple the approach on VIF (CV) and the one on SCR ($^{s.s}$) while estimating, so that the user can obtain optimised estimators. Then, we still consider *situation* β as the valuation time economic situation, so that we use notation $_\beta$ below (NPV_β/NPV_β^{CV}, $NPV_{\beta,i}/NPV_{\beta,i}^{CV,s.s}$, VIF_β/VIF_β^{CV}, $VIF_{\beta,i}/VIF_{\beta,i}^{CV,s.s}$, OF_β/OF_β^{CV}, $OF_{\beta,i}/OF_{\beta,i}^{CV,s.s}$, $SCR_\beta/SCR_\beta^{CV,s.s}$, $SCR_{\beta,i}/SCR_{\beta,i}^{CV,s.s}$, and more simply SR_β/SR_β^{CV}). Finally, we can define our optimal estimator, for each marginal risk i SCR,

$$\widehat{SCR}_{\beta,i}^{CV,s.s} = \widehat{OF}_\beta^{CV} - \widehat{OF}_{\beta,i}^{CV,s.s} \left(= \widehat{VIF}_\beta^{CV} - \widehat{VIF}_{\beta,i}^{CV,s.s} \right)$$

In all generality it seems very difficult to theoretically prove our *full-optimised* estimator $\widehat{SR}_\beta^{CV} = \frac{\widehat{OF}_\beta^{CV}}{\widehat{SCR}_\beta^{CV,s.s}}$ converges faster than the standard $\widehat{SR}_\beta = \frac{\widehat{OF}_\beta}{\widehat{SCR}_\beta}$. However, we have obtained some first results considering a marginal SCR as the solvency ratio denominator. This leads to a simpler/*marginal* solvency ratio $SR_{\beta,i} = \frac{OF_\beta}{SCR_{\beta,i}}$ (still considering risk i). We now compare the variances of an optimised $\left(\widehat{SR}_{\beta,i}^{CV} = \frac{\widehat{OF}_\beta^{CV}}{\widehat{SCR}_{\beta,i}^{CV,s.s}} \right)$ and a standard $\left(\widehat{SR}_{\beta,i} = \frac{\widehat{OF}_\beta}{\widehat{SCR}_{\beta,i}} \right)$ estimator.

Note first the three following points[11],

[11] We assume below, for the sake of simplicity, the aforementioned bias (see 4.3) impact is negligible.

$$\begin{cases} 1-\mathbb{V}\left[\widehat{OF}_{\beta}^{CV}\right]=\mathbb{V}\left[\widehat{VIF}_{\beta}^{CV}\right]=\frac{1}{S}\mathbb{V}\left[NPV_{\beta}(\varepsilon)+c^{*}\left(NPV_{\alpha}(\varepsilon)-\mathbb{E}\left[VIF_{\alpha}(\varepsilon)\right]\right)\right] \\ =\frac{1}{S}\left(\mathbb{V}\left[NPV_{\beta}(\varepsilon)\right]+c^{*}\mathbb{V}\left[NPV_{\alpha}(\varepsilon)\right]+2c^{*}.CoVar\left[NPV_{\alpha}(\varepsilon);NPV_{\beta}(\varepsilon)\right]\right)=\mathscr{O}\left(\frac{1}{S}\right), \\ 2-\mathbb{V}\left[\widehat{SCR}_{\beta,i}^{CV,s.s}\right]=\frac{1}{S}\mathbb{V}\left[NPV_{\beta}^{CV}(\varepsilon)-NPV_{\beta,i}^{CV,s.s}(\varepsilon)\right]=\mathscr{O}\left(\frac{1}{S}\right)\text{, and,} \\ 3-CoVar\left[\widehat{OF}_{\beta}^{CV};\widehat{SCR}_{\beta,i}^{CV,s.s}\right]=CoVar\left[\widehat{VIF}_{\beta}^{CV};\widehat{SCR}_{\beta,i}^{CV,s.s}\right] \\ =CoVar\left[\frac{1}{S}\sum_{s=1}^{S}NPV_{\beta}^{CV}(\varepsilon^{s});\frac{1}{S}\sum_{s=1}^{S}NPV_{\beta}^{CV}(\varepsilon^{s})-\frac{1}{S}\sum_{s=1}^{S}NPV_{\beta,i}^{CV,s.s}(\varepsilon^{s})\right] \\ =\frac{1}{S}\mathbb{V}\left[NPV_{\beta}^{CV}(\varepsilon)\right]-CoVar\left[\frac{1}{S}\sum_{s=1}^{S}NPV_{\beta}^{CV}(\varepsilon^{s});\frac{1}{S}\sum_{s=1}^{S}NPV_{\beta,i}^{CV,s.s}(\varepsilon^{s})\right] \\ =\frac{1}{S}\mathbb{V}\left[NPV_{\beta}^{CV}(\varepsilon)\right]-\frac{1}{S}CoVar\left[NPV_{\beta}^{CV}(\varepsilon);NPV_{\beta,i}^{CV,s.s}(\varepsilon)\right] \\ \text{so that,} \\ CoVar\left[\widehat{OF}_{\beta}^{CV};\widehat{SCR}_{\beta,i}^{CV,s.s}\right]=\mathscr{O}\left(\frac{1}{S}\right). \end{cases}$$

We can then use the well-known asymptotic delta-method approximation (see Stuart & Ord (1994), Elandt et al. (1980)) for the variance of our marginal solvency ratio,

$$\mathbb{V}\left[\widehat{SR}_{\beta,i}^{CV}\right]=\mathbb{V}\left[\frac{\widehat{OF}_{\beta}^{CV}}{\widehat{SCR}_{\beta,i}^{CV,s.s}}\right]=\mathbb{V}\left[\frac{\widehat{OF}_{\beta}^{CV}}{\widehat{OF}_{\beta}^{CV}-\widehat{OF}_{\beta,i}^{CV,s.s}}\right]=\mathbb{V}\left[\frac{\widehat{OF}_{\beta}^{CV}-\widehat{OF}_{\beta,i}^{CV,s.s}+\widehat{OF}_{\beta,i}^{CV,s.s}}{\widehat{OF}_{\beta}^{CV}-\widehat{OF}_{\beta,i}^{CV,s.s}}\right]=\mathbb{V}\left[\frac{\widehat{OF}_{\beta,i}^{CV,s.s}}{\widehat{OF}_{\beta}^{CV}-\widehat{OF}_{\beta,i}^{CV,s.s}}\right]$$

$$\approx\frac{1}{SCR_{\beta,i}^{2}}\mathbb{V}\left[\widehat{OF}_{\beta,i}^{CV,s.s}\right]+\frac{OF_{\beta,i}^{2}}{SCR_{\beta,i}^{3}}\mathbb{V}\left[\widehat{SCR}_{\beta,i}^{CV,s.s}\right]-2\frac{OF_{\beta,i}}{SCR_{\beta,i}^{3}}CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{SCR}_{\beta,i}^{CV,s.s}\right]$$

$$=\frac{1}{SCR_{\beta,i}^{2}}\mathbb{V}\left[\widehat{OF}_{\beta,i}^{CV,s.s}\right]+\frac{OF_{\beta,i}^{2}}{SCR_{\beta,i}^{3}}\mathbb{V}\left[\widehat{SCR}_{\beta,i}^{CV,s.s}\right]-2\frac{OF_{\beta,i}}{SCR_{\beta,i}^{3}}CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{OF}_{\beta}^{CV}-\widehat{OF}_{\beta,i}^{CV,s.s}\right]$$

With in particular,

$$CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{OF}_{\beta}^{CV}-\widehat{OF}_{\beta,i}^{CV,s.s}\right]=CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{OF}_{\beta}^{CV}\right]-\mathbb{V}\left[\widehat{OF}_{\beta,i}^{CV,s.s}\right]$$

so that,

$$\mathbb{V}\left[\widehat{SR}_{\beta,i}^{CV}\right]\approx\left(\frac{1}{SCR_{\beta,i}^{2}}+2\frac{OF_{\beta,i}}{SCR_{\beta,i}^{3}}\right)\mathbb{V}\left[\widehat{OF}_{\beta,i}^{CV,s.s}\right]+\frac{OF_{\beta,i}^{2}}{SCR_{\beta,i}^{3}}\mathbb{V}\left[\widehat{SCR}_{\beta,i}^{CV,s.s}\right]-2\frac{OF_{\beta,i}}{SCR_{\beta,i}^{3}}CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{OF}_{\beta}^{CV}\right].$$

Let us now assume (weak assumptions, from our work),

$$\begin{cases} \mathbb{V}\left[\widehat{OF}_{\beta,i}^{CV,s.s}\right]\leq\mathbb{V}\left[\widehat{OF}_{\beta,i}\right], \\ \mathbb{V}\left[\widehat{SCR}_{\beta,i}^{CV,s.s}\right]\leq\mathbb{V}\left[\widehat{SCR}_{\beta,i}\right], \\ CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{OF}_{\beta}^{CV}\right]>CoVar\left[\widehat{OF}_{\beta,i};\widehat{OF}_{\beta}\right]. \end{cases}$$

In particular, $CoVar\left[\widehat{OF}_{\beta,i}^{CV};\widehat{OF}_{\beta}^{CV}\right]\approx 0$[12]. We therefore obtain,

$$\left(\frac{1}{SCR_{\beta,i}^{2}}+2\frac{OF_{\beta,i}}{SCR_{\beta,i}^{3}}\right)\mathbb{V}\left[\widehat{OF}_{\beta,i}^{CV,s.s}\right]+\frac{OF_{\beta,i}^{2}}{SCR_{\beta,i}^{3}}\mathbb{V}\left[\widehat{SCR}_{\beta,i}^{CV,s.s}\right]-2\frac{OF_{\beta,i}}{SCR_{\beta,i}^{3}}CoVar\left[\widehat{OF}_{\beta,i}^{CV,s.s};\widehat{OF}_{\beta}^{CV}\right]$$

[12] The standard and shocked OF estimators being estimated using different simulation seed/economic scenarios, the respective NPV are independent and so is their average.

$$\leq \left(\frac{1}{SCR_{\beta,i}^2} + 2\frac{OF_{\beta,i}}{SCR_{\beta,i}^3}\right)\mathbb{V}\left[\widehat{OF}_{\beta,i}\right] + \frac{OF_{\beta,i}^2}{SCR_{\beta,i}^3}\mathbb{V}\left[\widehat{SCR}_{\beta,i}\right] - 2\frac{OF_{\beta,i}}{SCR_{\beta,i}^3}CoVar\left[\widehat{OF}_{\beta,i};\widehat{OF}_{\beta}\right].^{13}$$

We can finally assume (up to our delta method approximations and other weak assumptions) that, with our CV approach we are supposed to decrease the marginal solvency ratio estimator variance,

$$\mathbb{V}\left[\widehat{SR}_{\beta,i}^{CV}\right] \leq \mathbb{V}\left[\widehat{SR}_{\beta,i}\right].$$

Further developments (including at least empirical tests) will be required to better qualify our method impact on the solvency ratios precision.

[13] The approximation obtained using the delta method on $\widehat{SR}_{\beta,i}$.

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