Decomposing life insurance liabilities into risk factors

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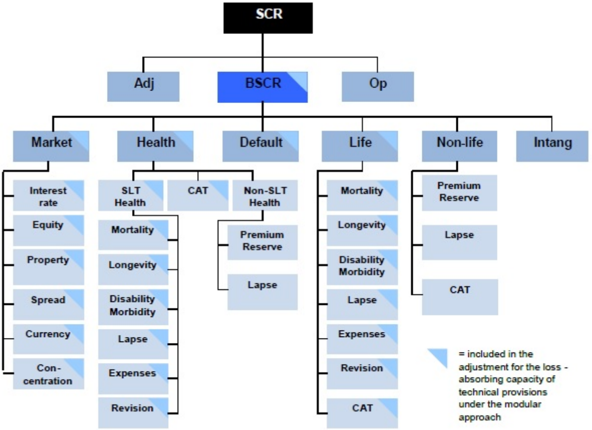
November 3, 2015

Life insurance risk and its sources



Source: *EIOPA-14-322 Underlying assumptions in the standard formula for the SCR calculation*

Additive decomposition of risk

Modeling Framework

Assume that the random variable $\bar{L} = L - \mathbf{E}[L]$ denotes the life insurer's total risk, which has k sources described by random variables $Z_1, \ldots, Z_k$.

- **typical approach:** given a risk measure ρ, find a decomposition $r_{Z_1}, \ldots, r_{Z_k} \in \mathbb{R}$ of the form

$$\rho(\bar{L}) = r_{Z_1} + \ldots + r_{Z_k}$$

- **our approach:** find random variables $R_{Z_1}, \ldots, R_{Z_k}$ such that

$$\bar{L} = R_{Z_1} + \ldots + R_{Z_k}$$

Why a random decomposition?

A **risk measure** $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$

- maps from a large space $\mathcal{L}$ to a small space $\mathbb{R} \cup \{+\infty\}$
- so it reduces complexity
- by deliberately discarding information.

A **random decomposition**

- keeps the complexity
- by retaining most of the information.

Are the different risk contributions $R_{Z_1}, \ldots, R_{Z_k}$

- *symmetric or skewed,*
- *heavy-tailed or light-tailed,*
- *... ?*

Random decompositions in the literature

Notation

Let $(R_1, \ldots, R_k)$ be a random decomposition of $\bar{L} = L - \mathbf{E}[L]$ with respect to $Z = (Z_1, \ldots, Z_k)$, where R_i corresponds to risk source Z_i, $i = 1, \ldots, k$.

(a) orthogonalization by conditional expectations

e.g. *Bühlmann (1995), Fischer (2004), C. & Helwich (2008)*

For $Z = (Z_1, Z_2)$ set

$$\bar{L} = \underbrace{\mathbf{E}[\bar{L}|Z_1]}_{=R_1} + \underbrace{\bar{L} - \mathbf{E}[\bar{L}|Z_1]}_{=R_2}$$

Random decompositions in the literature

(b) Hoeffding decomposition

e.g. *Rosen & Saunders (2010)*

For $Z = (Z_1, Z_2)$ set

$$\bar{L} = \underbrace{\mathbf{E}[\bar{L}|Z_1]}_{=R_1} + \underbrace{\mathbf{E}[\bar{L}|Z_2]}_{=R_2} + \underbrace{\mathbf{E}[\bar{L}|Z_1, Z_2] - \mathbf{E}[\bar{L}|Z_1] - \mathbf{E}[\bar{L}|Z_2]}_{=R_{1,2}}$$

(c) Taylor expansion

e.g. *C. (2007)*

For $\bar{L} = F(Z_1, Z_2)$ and arbitrary but fixed $(z_1, z_2) \in \mathbf{R}^2$ set

$$\bar{L} \approx F(z_1, z_2) + \underbrace{(Z_1 - z_1)\frac{\partial F}{\partial z_1}(z_1, z_2)}_{=R_1} + \underbrace{(Z_2 - z_2)\frac{\partial F}{\partial z_2}(z_1, z_2)}_{=R_2}$$

Random decompositions in the literature

(d) Solvency II approach

e.g. *CEIOPS (2010), Artinger (2010), Gatzert & Wesker (2012)*

For $\bar{L} = F(Z_1, Z_2)$ and arbitrary but fixed $(z_1, z_2) \in \mathbf{R}^2$ set

$$\bar{L} \overset{?}{\approx} \underbrace{F(Z_1, z_2)}_{=R_1} + \underbrace{F(z_1, Z_2)}_{=R_2}$$

Meaningful risk decompositions

- P1 **Randomness:** For $(\bar{L}, Z_1, \ldots, Z_k)$ find a random vector $(R_1, \ldots, R_k)$. We write $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (R_1, \ldots, R_k)$.
- P2 **Attribution:** If $\bar{L}$ is $\sigma(Z_i)$-measurable and Z_i is independent of $(Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k)$, then we should have $R_j = 0$ for all $j \neq i$.
- P3 **Uniqueness:** We want that $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (R_1, \ldots, R_k)$ and $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (\tilde{R}_1, \ldots, \tilde{R}_k)$ implies $R_i = \tilde{R}_i$ for all $i = 1, \ldots, k$.
- P4 **Order invariance:** For $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (R_1, \ldots, R_k)$ and any permutation $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ we want that $(\bar{L}, Z_{\pi(1)}, \ldots, Z_{\pi(k)}) \leftrightarrow (R_{\pi(1)}, \ldots, R_{\pi(k)})$.

Meaningful risk decompositions

- P5 **Scale invariance:** We want that $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (R_1, \ldots, R_k)$, $(\bar{L}, \tilde{Z}_1, \ldots, \tilde{Z}_k) \leftrightarrow (\tilde{R}_1, \ldots, \tilde{R}_k)$ and $\tilde{Z}_i = f_i(Z_i)$ for all i for invertible functions f_i implies $R_i = \tilde{R}_i$ for all $i = 1, \ldots, k$.

- P6* **Additive Aggregation:** For $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (R_1, \ldots, R_k)$ we want that $\bar{L} = R_1 + \ldots + R_k$.

- P6 **Aggregation:** For $(\bar{L}, Z_1, \ldots, Z_k) \leftrightarrow (R_1, \ldots, R_k)$ we want that $\bar{L} = F(R_1, \ldots, R_k)$ for some function $F : \mathbf{R}^k \to \mathbf{R}$.

Meaningful risk decompositions

	P1	P2	P3	P4	P5	P6	P6*
orthog. by cond. exp.	✓	✓	✓	×	✓	✓	✓
Hoeffding decomp.	✓	✓	✓	✓	✓	×	×
Taylor expansion	✓	✓	×	✓	×	×	×
Solvency II approach	✓	×	×	✓	✓	×	×
MRT decomposition	✓	✓	✓	✓	✓	✓	✓

P1 - Randomness
P2 - Attribution
P3 - Uniqueness
P4 - Order invariance
P5 - Scale invariance
P6 - Aggregation
P6* - Additive Aggregation

The MRT decomposition – Concept Overview

- Express all risk sources as stochastic processes $Z_1(t), \dots, Z_k(t)$, $t \in [0, T]$
- Calculate the corresponding martingale parts $M_{Z_1}(t), \dots, M_{Z_k}(t)$, $t \in [0, T]$
- Decompose $\bar{L} = L - \mathbf{E}[L]$ with the Martingale Representation Theorem to

$$\bar{L} = \int_0^T \psi_1(t) \, \mathrm{d}M_{Z_1}(t) + \ldots + \int_0^T \psi_k(t) \, \mathrm{d}M_{Z_k}(t)$$

- Interpret the resulting addends as risk contributions

$$R_i = \int_0^T \psi_i(t) \, \mathrm{d}M_{Z_i}(t), \quad i = 1, \dots, k$$

The MRT decomposition – Modeling Framework

- All **financial and demographic risk factors** are modelled by a $(k-1)$-dimensional Ito process

$$\mathrm{d}X(t) = \theta(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \quad t \in [0, T], \quad X(0) = x_0 \in \mathbf{R}^{k-1},$$

  where W is a d-dimensional Brownian motion with natural filtration $(\mathcal{G}_t)_t$.
- The **portfolio survivor status** is a doubly stochastic counting process $m - N(t)$ with $(\mathcal{G}_t)_t$-predictable jump intensity $(\mu(t))_t$, where $m \in \mathbf{N}$ is the initial size of the insurance portfolio.

Conditional on $\mathcal{G}_T$, remaining lifetimes shall be independent.

The MRT decomposition – Modeling Framework

The insurer's total net liability at time 0 is

$$L = C + \underbrace{\sum_{t_i} (m - N(t_i)) C_{a,i}}_{\text{discrete survival payments}} + \underbrace{\int_0^T (m - N(s)) C_a(s) \mathrm{d}s}_{\text{continuous survival payments}}$$

$$+ \underbrace{\sum_{t_i} (N(t_i) - N(t_{i-1})) C_{ad,i}}_{\text{discrete death benefits}} + \underbrace{\int_0^T C_{ad}(s) \mathrm{d}N(s)}_{\text{continuous death benefits}}$$

where C, $C_{a,i}$, $C_{ad,i}$ are $\mathcal{G}_T$-measurable and $C_a(t)$, $C_{ad}(t)$ are $\mathcal{G}_t$-measurable

Proposition (Martingale Representation Theorem)

Let L be square integrable. If $X(t)$ has a unique strong solution, $k-1=d$, $\det\sigma(t)\neq 0$ for all t, and L is integrable, then there exist unique, predictable processes $\psi_1^W(t),\dots,\psi_{k-1}^W(t),\psi^N(t)$ such that

$$\bar{L} = L - \mathbf{E}[L] = \sum_{i=1}^{k-1} \underbrace{\int_0^T \psi_i^W(t)\mathrm{d}M_i^W(t)}_{R_i} + \underbrace{\int_0^T \psi^N(t)\mathrm{d}M^N(t)}_{R_k},$$

$$M_i^W(t) = \sum_{j=1}^{d}\int_0^t \sigma_{ij}(s)\mathrm{d}W_j(s),$$

$$M^N(t) = N(t) - \int_0^t (m - N(s-))\mu(s)\mathrm{d}s.$$

Definition (MRT decomposition)

We call $(R_1,\dots,R_k)$ the MRT decomposition of $(\bar{L},X_1,\dots,X_{k-1},N)$.

The MRT decomposition – Calculation

How can we find the processes $\psi_1^W(t), \dots, \psi_{k-1}^W(t), \psi^N(t)$?

- Use the **Clark-Ocone formula** from Malliavin calculus?
  Driving processes need to be independent!
  Not true for $N(t)$ and $X(t)$!
- Use results from **BSDE theory**?
  Most results in the literature are not constructive.

Sketch of our solution

- Apply Clark-Ocone formula on $\mathbf{E}[\bar{L}|\mathcal{G}_t]$
- Add the $\mathrm{d}M^N(t)$ part by explicit calculation (non trivial!)
- Transform integrator $\mathrm{d}W(t)$ to $\mathrm{d}M^W(t)$ under the assumptions $k-1=d$ and $\det \sigma(t) \neq 0$ for all t

The MRT decomposition – Calculation

Additionally **assume** that $X(t)$ is **Markovian.**

How can we then find $\psi_1^W(t), \ldots, \psi_{k-1}^W(t), \psi^N(t)$?

- Use **Ito's formula** under differentiability assumptions.

Sketch of our solution

- Assume that C, $C_{a,i}$, $C_{ad,i}$ are of the form $e^{-\int_0^T g(s,X(s))\mathrm{d}s} h(X(T))$
- Assume that $C_a(t)$, $C_{ad}(t)$ are of the form $e^{-\int_0^t g(s,X(s))\mathrm{d}s} h(X(t))$
- Assume that $f(t,x) = \mathbf{E}[\bar{L}|X(t) = x]$ has continuous derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$
- Apply the Feynman-Kac formula

The MRT decomposition – Decomposition Properties

$$\bar{L} = \sum_{i=1}^{k-1} \underbrace{\int_0^T \psi_i^W(t) \mathrm{d}M_i^W(t)}_{R_i} + \underbrace{\int_0^T \psi^N(t) \mathrm{d}M^N(t)}_{R_k}$$

- P1 Randomness: ✓ (obvious)
- P2 Attribution: ✓ (non trivial)
- P3 Uniqueness: ✓ (see uniqueness in MRT)
- P4 Order invariance: ✓ (use uniqueness in MRT)
- P5 Scale invariance: ✓ (use Ito's Lemma)
  given that the scaling functions
  - $f_1, \dots, f_{k-1}$ are twice continuously differentiable
  - $f_k(N(t))$ is again a counting process
- P6 Aggregation: ✓ (obvious)
- P6* Additive Aggregation: ✓ (obvious)

The MRT decomposition – Diversification Properties

For portfolio size $m \to \infty$, what happens with the average risk

$$\frac{1}{m}\bar{L} \overset{\text{(P5)}}{=} \frac{1}{m}R_1 + \ldots + \frac{1}{m}R_k \longrightarrow \quad ?$$

Proposition (diversification of unsystematic risk)

Let $\mathbf{E}[\mu(t)^4]$, $C_{a,i}$, $C_a(t)$, $C_{ad,i}$, and $C_{ad}(t)$ be bounded. Then

$$\frac{1}{m}R_k = \frac{1}{m}\int_0^T \psi^N(t)\mathrm{d}M^N(t) \xrightarrow{L_2} 0, \quad m \to \infty.$$

The MRT decomposition – Diversification Properties

The *diversified* total risk per policy is given by the limit

$$\lim_{m\to\infty} \frac{1}{m}\bar{L}^{(m)} \stackrel{P}{=} \mathbf{E}[\bar{L}^{(1)}|\mathcal{G}_T].$$

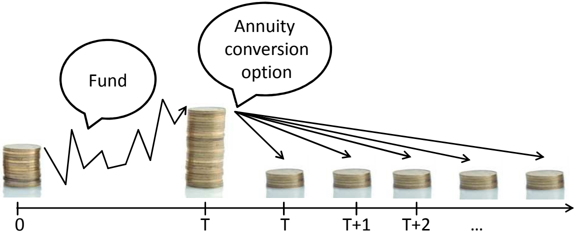
Proposition (diversification limit of MRT decomposition)

Define the MRT decompositions

- $(\frac{1}{m}\bar{L}^{(m)}, X, N) \leftrightarrow (R_1^{(m)}, \ldots, R_k^{(m)})$ for each portfolio size m,
- $(\mathbf{E}[\bar{L}^{(1)}|\mathcal{G}_T], X) \leftrightarrow (R_1^*, \ldots, R_{k-1}^*)$.

Then $(R_1^{(m)}, \ldots, R_{k-1}^{(m)}) \xrightarrow{P} (R_1^*, \ldots, R_{k-1}^*), \quad m \to \infty$.

Example: Variable Annuity with GMDB



- single premium $P_0 = 1$ at time zero with age $x = 50$
- savings phase of $T = 15$ years
- policyholder's account value is $A(t) = P_0 \frac{S(t)}{S(0)}$ at time t
- **GMDB** (guaranteed minimum death benefit) of size $\max\{P_0, A(\lceil t \rceil)\}$ at time $\lceil t \rceil$ for death at time t

- **Fund**: Geometric Brownian motion

$$dS(t) = \mu_S S(t)dt + \sigma_S S(t)dW_S(t), \; S(0) > 0.$$

- **Interest**: Cox-Ingersoll-Ross model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r \sqrt{r(t)}dW_r(t), \; r(0) > 0.$$

- **Systematic mortality**: time-inhomogeneous Cox-Ingersoll-Ross model

$$d\mu(t,x) = (\gamma(t,x) - \delta(t,x)\mu(t,x))dt + \sigma_u(t,x)dW_\mu(t), \quad \mu(0,x) > 0.$$

- **Unsystematic mortality**:
  $N(t)$ is binomially distributed given $(\mu(s))_{0 \le s \le t}$

Assumption: Processes $S, \; r$ and μ are independent.

- **number of policies** $m = 100$

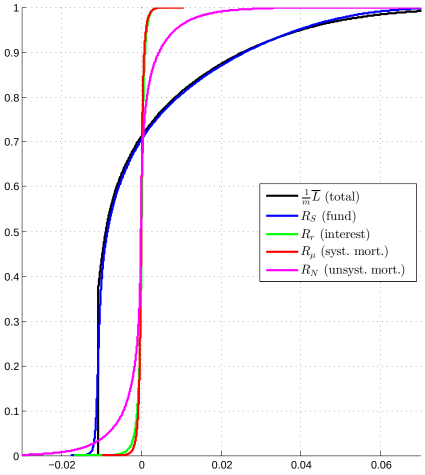
total loss

$$L = \sum_{t=1}^{T} (N(t) - N(t-1)) e^{-\int_0^t r(s)\mathrm{d}s} \max\{P_0 - A(t), 0\}$$

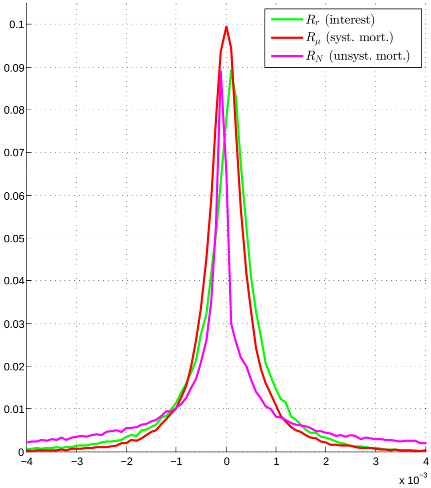
MRT decomposition

$$(\frac{1}{m}\bar{L}, S, r, \mu, N) \leftrightarrow (R_S, R_r, R_\mu, R_N)$$

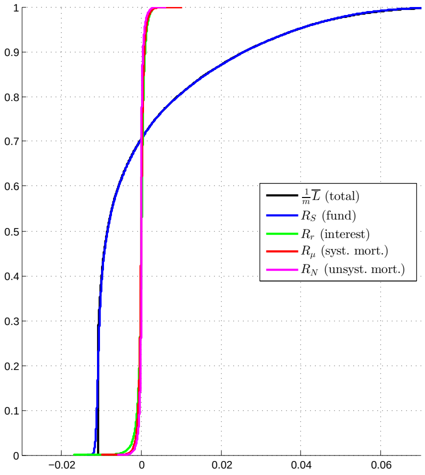
satisfying $\frac{1}{m}\bar{L} = R_S + R_r + R_\mu + R_N$



cumulative distributions functions of $\frac{1}{m}\overline{L}$, R_S, R_r, R_μ, R_N



density functions of R_r, R_μ, R_N



cumulative distributions functions of $\frac{1}{m}\overline{L}$, R_S, R_r, R_μ, R_N
portfolio size increased to $m = 10000$

Conclusion

- *random* decompositions reflect complexity of total risk
- our MRT decomposition has numerous *nice properties*

For further details, please see

- Schilling, K., Bauer, D., Christiansen, M.C., Kling, A., (2015). Decomposing life insurance liabilities into risk factors. *Preprint Series 2015/01, Ulm University*.