Discussion sur
"Decomposing life insurance liabilities into risk factors"
par M. Christansen & al

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Life insurance policy issued at time 0 and terminating at time T
State of policy at time t is $Z_t \in \mathbb{Z} = \{0, \ldots, n\}$, $Z_0 = 0$,
$\mathbf{F}^Z = (\mathcal{F}_t^Z)_{t \in [0,T]}$
Payments (benefits less premiums) in $[0, t]$ total B_t:

$$dB_t = \sum_j I_{t-}^j \, dB_t^j \, dt + \sum_{j \neq k} b_t^{jk} \, dN_t^{jk}$$

$I_t^j = 1_{[Z_t = j]}$, state indicators
$N_t^{jk} = \sharp\{s;\ s \leq t,\ Z_{s-} = j,\ Z_s = k\}$, $j \neq k$, counting processes
B^j annuity payable in state j at time t, predictable
b^{jk} sum assured payable upon transition $j \to k$, predictable
Intensities of transition: $I_t^j \, \mu_t^{jk}$, $j \neq k$
Rate of interest at time t: r_t

State of economic-demographic environment at time t is $Y_t \in \mathbb{Y} = \{1, \ldots, m\}$,
$\mathbf{F}^Y = (\mathcal{F}_t^Y)_{t \in [0,T]}$
$\hat{I}_t^e = 1_{[Y_t = e]}$, state indicators
$\hat{N}_t^{ef} = \sharp\{s;\ s \leq t,\ Y_{s-} = e,\ Y_s = f\}$, $e \neq f$, counting processes

The contract should be “fair”, which (at least) must mean

$$\mathbb{E}\left[\int_0^T e^{-\int_0^\tau r} dB_\tau\right] = 0 \tag{1}$$

In life insurance one used to require more. “Principle of equivalence”:

$$\mathbb{E}\left[\int_0^T e^{-\int_0^\tau r} dB_\tau \,\middle|\, \mathcal{F}_T^Y\right] = 0 \tag{2}$$

or

$$\int_0^T e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau)\left(b_\tau^j + \sum_{k;k\neq j} b_\tau^{jk}\,\mu_\tau^{jk}\right) d\tau = 0 \tag{3}$$

Thus, B must be adapted not only to $\mathbf{F}^Z$, but to $\mathbf{F}^Y \vee \mathbf{F}^Z$.

Motivation: The idea of insurance is to average out purely random differences between policy histories.

Consider portfolio of K identical replicates of the generic policy:

$$\frac{1}{K}\sum_{k=1}^{K}\int_0^T e^{-\int_0^\tau r}\,dB_\tau^{(k)} \to \mathbb{E}\left[\int_0^T e^{-\int_0^\tau r}\,dB_\tau \,\middle|\, \mathcal{F}_T^Y\right] \tag{4}$$

Equivalence implies

$$\mathbb{E}\left[\int_0^T e^{-\int_0^\tau r}dB_\tau \,\middle|\, \mathcal{F}_t^Y\right] = 0$$

or

$$\mathbb{E}\left[\int_0^t e^{\int_\tau^t r}d(-B)_\tau \,\middle|\, \mathcal{F}_t^Y\right] = \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r}dB_\tau \,\middle|\, \mathcal{F}_t^Y\right]$$

On the average in the portfolio, retrospective reserve equals prospective reserve at all times.

Solvency risk is only due to "idiosyncratic risk", i.e. purely random deviation of individual policy histories from (conditionally) expected.

$$\mathbb{V}\mathrm{ar}\left[\frac{1}{K}\sum_{k=1}^K \int_0^T e^{-\int_0^\tau r}dB_\tau^{(k)}\right] = \frac{1}{K}\mathbb{E}\,\mathbb{V}\mathrm{ar}\left[\int_0^T e^{-\int_0^\tau r}dB_\tau \,|\, \mathcal{F}_T^Y\right] \to 0$$

WITH PROFIT

Payments $\bar{b}_t^j$ and $\bar{b}_t^{jk}$ *guaranteed* at time 0. Designed by equivalence principle using prudent *technical basis* with deterministic elements $\bar{r}_t$ and $\bar{\mu}_t^{jk}$:

$$\int_0^T e^{-\int_0^\tau \bar{r}} \sum_j \bar{p}^{0j}(0,\tau) \left(b_\tau^j + \sum_{k;k\neq j} b_\tau^{jk}\, \bar{\mu}_\tau^{jk} \right) d\tau = 0 \qquad (5)$$

Prospective technical reserve in state j at time t:

$$\bar{V}_t^j = \bar{\mathbb{E}}\left[\int_t^T e^{-\int_t^\tau \bar{r}} dB_\tau \,\middle|\, Z_t = j\right]$$
$$= \int_t^T e^{-\int_t^\tau \bar{r}} \sum_g \bar{p}^{jg}(t,\tau)\left(b_\tau^g + \sum_{h;h\neq g} b_\tau^{gh}\bar{\mu}_\tau^{gh}\right) d\tau \qquad (6)$$

Reserve (insurer's debt to insured) should be non-negative for any sensible insurance product. Equivalence $\bar{V}_0^0 = 0$.

Thiele's differential equations

$$\frac{d}{dt}\bar{V}_t^j = \bar{r}_t \bar{V}_t^j - b_t^j - \sum_{k;\,k\neq j} \bar{R}_t^{jk}\,\bar{\mu}_t^{jk} \qquad (7)$$

$$\bar{R}_t^{jk} = b_t^{jk} + \bar{V}_t^k - \bar{V}_t^j \quad \text{(sum at risk)} \qquad (8)$$

Technical surplus by time t is retrospective reserve based on realized elements less prospective reserve based on technical elements:

$$S_t = -e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(\bar{b}^j_\tau + \sum_{k;k\neq j} \bar{b}^{jk}_\tau \, \mu^{jk}_\tau \right) d\tau - \sum_j p^{0j}(0,t) \, \bar{V}^j_t$$

Differentiate using Thiele for the $\bar{V}^j_t$ and Kolmogorov forward for the $p^{0j}(0,t)$,

$$\frac{d}{dt} p^{0j}(0,t) = \sum_{g;g\neq j} p^{0g}(0,t) \, \mu^{gj}_t - p^{0j}(0,t) \sum_{g;g\neq j} \mu^{jg}_t$$

$$dS_t = S_t\, r_t\, dt + \sum_j p^{0j}(0,t)\, c_t^j\, dt \tag{9}$$

where c_t^j is the rate at which technical surplus emerges in state j at time t

$$c_t^j = \bar{V}_t^j(r_t - \bar{r}_t) + \sum_{k;\, k \neq j} \bar{R}_t^{jk}\, (\bar{\mu}_t^{jk} - \mu_t^{jk}) \tag{10}$$

Technical interest to the safe side if $\bar{r}_t \leq r_t$.
Technical rate of transition $j \to k$ to the safe side if $\mathrm{sign}(\bar{\mu}_t^{jk} - \mu_t^{jk}) = \mathrm{sign}\bar{R}_t^{jk}$.

Integrate (9), using side condition $S_0 = 0$, to recast

$$S_t = e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau)\, c_\tau^j\, d\tau \tag{11}$$

Technical surpluses are to be paid back as *bonuses*, assume here as annuity dividends at rate δ_t^j in state j at time t and assurance dividends δ_t^{jk} upon transition $j \to k$ at time t.
The *net surplus* at time t is

$$
\begin{aligned}
W_t &= S_t - e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(\delta_\tau^j + \sum_{k;k\neq j} \delta_\tau^{jk} \, \mu_\tau^{jk} \right) d\tau \quad (12) \\
&= e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(c_\tau^j - \delta_\tau^j - \sum_{k;k\neq j} \delta_\tau^{jk} \, \mu_\tau^{jk} \right) d\tau \quad (13) \\
& \qquad (14)
\end{aligned}
$$

This is the balance of the account after accumulated dividends have been deducted from the technical surplus (11).

Dividends are not stipulated in the contract: they are controlled by the pension fund/insurance company.
They need to be non-negative and to satisfy

$$W_t \geq 0, \ \ \forall t$$

and $W_T = 0$, which means equivalence reestablished with factual rates:

$$\int_0^T e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(\bar{b}_\tau^j + d_\tau^j + \sum_{k;k\neq j} (\bar{b}_\tau^{jk} + \delta_\tau^{jk})\, \mu_\tau^{jk} \right) d\tau = 0$$

Remark 1: No model assumptions for r and μ^{jk}

Remark 2: Solvency for sure: No "longevity risk" (if technical elements chosen with sufficient prudence)

PERFECT INDEX-LINKED INSURANCE or "AUTOMATIC BALANCING MECHANISM (ABM)"

Policy issued at time 0 specifies baseline rates $\bar{r}_t$, $\bar{\mu}_t^{jk}$ that are "best estimates" of future rates, baseline payments $\bar{b}_t^j$, $\bar{b}_t^{jk}$ that represent target premiums and benefits under the baseline scenario, and contractual payments b_t^j and b_t^{jk} that are adapted to the realized indices r_t, μ_t^{jk} as follows:

$$b_t^j = \frac{e^{-\int_0^t \bar{r}}\, \bar{p}^{0j}(0,t)}{e^{-\int_0^t r}\, p^{0j}(0,t)}\ \bar{b}_t^j\ , \qquad b_t^{jk} = \frac{e^{-\int_0^t \bar{r}}\, \bar{p}^{0j}(0,t)\, \bar{\mu}_t^{jk}}{e^{-\int_0^t r}\, p^{0j}(0,t)\, \mu_t^{jk}}\ \bar{b}_t^{jk}$$

Thus, contractual payments are not specified in nominal terms (USD, GBP, EURO,...): they are index regulated so as to follow inflation (to the extent it is linked to interest) and the development of mortality.

The expected discounted payment (benefits less premiums) under the realized economic-demographic scenario is

$$\int_0^T e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(b_\tau^j + \sum_{k;k\neq j} b_\tau^{jk}\, \mu_\tau^{jk} \right) d\tau$$

$$= \int_0^T e^{-\int_0^\tau \bar{r}} \sum_j \bar{p}^{0j}(0,\tau) \left(\bar{b}_\tau^j + \sum_{k;k\neq j} \bar{b}_\tau^{jk}\, \bar{\mu}_\tau^{jk} \right) d\tau$$

which can be arranged to be 0 by choice of baseline elements satisfying "baseline equivalence".

No model assumptions about future rates over the next some 50 years. No solvency risk due to uncertain future rates. The insured has to live - and die - with what the factual rates can sustain.

EXAMPLE: LIFE ENDOWMENT

Classical: Single premium (price) of life endowment with sum b:

$$\pi = e^{-\int_0^T (r_u + \mu_{x+u})\, du}\, b$$

Problem if future r and μ unknown at time 0. ($\pi = \mathbb{E}\left[e^{-\int_0^T (r_u + \mu_{x+u})\, du}\right] b$?)

Solution is conditional equivalence: Sum assured for variable interest and mortality endowment with fixed premium π:

$$b = e^{\int_0^T (r_u + \mu_{x+u})\, du} \pi$$

Conditional equivalence with premium π:

$$e^{-\int_0^T (r_u + \mu_{x+u})\, du}\, b - \pi = 0$$

for sure. No interest risk, no mortality risk.

GUARANTEE: MGRI life endowment has sum assured

$$b^G = e^{\int_0^T (r_u \vee \underline{r}) + \mu_{x+u})\, du} \pi$$

Since $b^G > b$, the guarantee means the customer must purchase more insurance, only with a different design. It must be paid for with $\pi' > \pi$

Conditional expected equivalence is impossible since

$$e^{-\int_0^T (r_u + \mu_{x+u})\, du}\, b^G - \pi' = e^{\int_0^T (\underline{r} - r_u)^+ \, du} \pi - \pi'$$

is random variable. The guarantee reintroduces risk.

The price π' must be determined by fairness principle:

$$\mathbb{E}\, e^{\int_0^T (\underline{r} - r_u)^+ \, du} \pi - \pi' = 0$$

The risk can be measured by the variance. Essentially

$$\mathbb{V}\mathrm{ar}\, e^{\int_0^T (\underline{r}-r_u)^+ \, du}$$

Assume Y is Markov, and $r_t = r^{Y_t} = \sum_e \widehat{I}_t^e \, r^e$.

Martingale technique:

$$\begin{aligned} M_t &= \mathbb{E}[e^{\int_0^T (\underline{r}-r_u)^+ \, du} \mid \mathcal{F}_t^Y] \\ &= e^{\int_0^t (\underline{r}-r_u)^+ \, du} \mathbb{E}[e^{\int_t^T (\underline{r}-r_u)^+ \, du} \mid \mathcal{F}_t^Y] \\ &= e^{\int_0^t (\underline{r}-r_u)^+ \, du} \sum_e \widehat{I}_t^e \, v^e(t) \end{aligned}$$

$$v^e(t) = \mathbb{E}[e^{\int_t^T (\underline{r}-r_u)^+ \, du} \mid Y_t = e]$$

$$
\begin{aligned}
dM_t &= d\left[e^{\int_0^t (\underline{r}-r_u)^+\,du}\sum_e \widehat{I}_t^e\, v^e(t)\right] \\
&= e^{\int_0^t (\underline{r}-r_u)^+\,du}(\underline{r}-r_t)^+ dt \sum_e \widehat{I}_t^e\, v^e(t) \\
&\quad + e^{\int_0^t (\underline{r}-r_u)^+\,du}\sum_{e\neq f} d\widehat{N}_t^{ef}\,(v^f(t)-v^e(t)) + e^{\int_0^t (\underline{r}-r_u)^+\,du}\sum_e \widehat{I}_t^e \frac{d}{dt}v^e(t)\,dt \\
&= e^{\int_0^t (\underline{r}-r_u)^+ du}\sum_e \widehat{I}_t^e\left((\underline{r}-r^e)^+ v^e(t) + \sum_{f;f\neq e}\lambda^{ef}(v^f(t)-v^e(t))\frac{d}{dt}v^e(t)\right)dt \\
&\quad + e^{\int_0^t (\underline{r}-r_u)^+\,du}\sum_{e\neq f}(v^f(t)-v^e(t))\,d\widehat{M}_t^{ef}
\end{aligned}
$$

$d\widehat{M}_t^{ef} = d\widehat{N}_t^{ef} - \widehat{I}_t^e\lambda^{ef}\,dt$, orthogonal martingales.

Drift part must be null. Gives the constructive differential equations

$$(\underline{r} - r^e)^+ v^e(t) + \sum_{f;f\neq e} \lambda^{ef} \left(v^f(t) - v^e(t)\right) + \frac{d}{dt} v^e(t) = 0$$

with boundary conditions $v^e(T) = 1$. Matrix form

$$\frac{d}{dt}\mathbf{v}(t) - \left(\mathbf{D}\mathrm{iag}((\underline{r} - r^e)^+) + \mathbf{\Lambda}\right) \mathbf{v}t = \mathbf{0}\,,$$

and condition $\mathbf{v}(T) = \mathbf{1}$. Solution:

$$\mathbf{v}(t) = \exp\{(\mathbf{D}\mathrm{iag}((\underline{r} - r^e)^+) + \tilde{\mathbf{\Lambda}})\,(T-t)\}\mathbf{1}\,. \tag{15}$$

$$dM_t = e^{\int_0^t (\underline{r} - r_u)^+ \, du} \sum_{e\neq f} \left(v^f(t) - v^e(t)\right) d\widehat{M}_t^{ef}$$

Risk:

$$
\begin{aligned}
R &= \mathbb{V}\mathrm{ar}\left[e^{\int_0^T (\underline{r}-r_u)^+\,du}\right] \\
&= \mathbb{V}\mathrm{ar} M_T \\
&= \mathbb{E}\left[\sum_e \int_0^T e^{2\int_0^s (\underline{r}-r_u)^+\,du} \sum_{f;f\neq e} (v^f(s)-v^e(s))^2\, \widehat{I}_s^e\, \lambda^{ef}\, ds\right]
\end{aligned}
$$

Martingale technique again: Start from martingale

$$M_R(t) = \mathbb{E}\left[\sum_e \int_0^T e^{2\int_0^s (\underline{r}-r_u)^+\,du} \sum_{f;f\neq e} (v^f(s)-v^e(s))^2\, \widehat{I}_s^e\, \lambda^{ef}\, ds \,|\, \mathcal{F}_t^Y\right]$$

$$= \int_0^t e^{2\int_0^s (\underline{r}-r_u)^+\,du} \sum_e \widehat{I}_s^e \sum_{f;f\neq e} (v^f(s)-v^e(s))^2\, \lambda^{ef}\, ds$$

$$+ e^{2\int_0^t (\underline{r}-r_u)^+ du}\, \mathbb{E}\left[\sum_e \widehat{I}_s^e \int_t^T e^{2\int_t^s (\underline{r}-r_u)^+ du} \sum_{f;f\neq e} (v^f(s)-v^e(s))^2\, \lambda^{ef}\, ds \,|\, \mathcal{F}_t^Y\right]$$

$$= \int_0^t e^{2\int_0^s (\underline{r}-r_u)^+ du} \sum_e \widehat{I}_s^e \sum_{f\neq e} (v^f(s)-v^e(s))^2 \lambda^{ef} ds + e^{2\int_0^t (\underline{r}-r_u)^+ du} \sum_g \widehat{I}_t^e w^g(t)$$

Calculate $dM_R(t)$, solve the $w^g(t)$ from differential equations obtained from setting drift part to 0, subject to side conditions $w^g(T) = 0$, and finally obtain $R = w^{Y_0}(0)$.

In general use the decomposition;

$$\int_0^T e^{-\int_0^\tau r} dB_\tau = \sum_{j \neq k} \int_0^T \xi_s^{jk} \, dM_s^{jk} + \sum_{e \neq f} \int_0^T \eta_s^{ef} \, dM_s^{ef}$$

Orthogonal martingales:

$$\begin{aligned} R &= \mathbb{V}\text{ar}[\int_0^T e^{-\int_0^\tau r} dB_\tau] \\ &= \sum_{j \neq k} \mathbb{E}\left[\int_0^T (\xi_t^{jk})^2 \, I_s^j \mu_s^{jk} \, ds\right] + \sum_{e \neq f} \mathbb{E}\left[\int_0^T (\eta_s^{jk})^2 \, \widehat{I}_s^j \lambda_s^{ef} \, ds\right] \\ &= \sum_{j \neq k} R^{jk} + \sum_{e \neq f} \widehat{R}^{ef} \end{aligned}$$

The risk components are *numbers*, not random variables.

The R^{jk} represent idiosyncratic risk that has to be there. Negligible in large portfolio.

The R^{ef} represent environment risk that does not have to – and really should not – be there. It is created by the design of the product (guarantees etc). Disappears if conditional equivalence is imposed.

To the ACPR: Use your Authority, Control the products, impose Prudence, perform strict Regulation
in the interest of solvency and transparency.